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The dynamics of strong turbulence at free surfaces. Part 2. Free-surface boundary conditions

By M. BROCCHINI[1] AND D. H. PEREGRINE[2]

[1]D.I.Am., Università di Genova, 16145 Genova, Italy

[2]School of Mathematics, University of Bristol, Bristol BS8 1TW, UK

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Strong turbulence at a water–air free surface can lead to splashing and a disconnected surface as in a breaking wave. Averaging to obtain boundary conditions for such flows first requires equations of motion for the two-phase region. These are derived using an integral method, then averaged conservation equations for mass and momentum are obtained along with an equation for the turbulent kinetic energy in which extra work terms appear. These extra terms include both the mean pressure and the mean rate of strain and have similarities to those for a compressible fluid. Boundary conditions appropriate for use with averaged equations in the body of the water are obtained by integrating across the two-phase surface layer.

A number of 'new' terms arise for which closure expressions must be found for practical use. Our knowledge of the properties of strong turbulence at a free surface is insufficient to make such closures. However, preliminary discussions are given for two simplified cases in order to stimulate further experimental and theoretical studies.

Much of the turbulence in a spilling breaker originates from its foot where turbulent water meets undisturbed water. A discussion of averaging at the foot of a breaker gives parameters that may serve to measure the 'strength' of a breaker.

1. Introduction

In Part 1 (Brocchini & Peregrine 2001) the effects of strong turbulence at a free surface are described, and a semi-quantitative classification introduced. Part 2 is also motivated by a wish to improve modelling of breaking water waves, but the analysis is applicable to the wider range of flows described in Part 1. We approach the problem from the standard Reynolds averaging of the equations of motion of a turbulent fluid. The two-phase nature of the surface region, especially when splashing occurs, is recognized. We choose to derive anew the equations of motion for the two-phase flow in order to better display effects of interest and to introduce notation. Although the equations are not new, our derivation using integral formulations is new and leads to some extra terms in the transport equation for the turbulent kinetic energy. However, the main aim is to obtain a simplification leading to a form of boundary conditions that can be applied to averaged equations of motion for the main volume of water. Although we usually refer to air and water the analysis is applicable to any combination of liquid and gas which have a similar large ratio of densities and at scales where viscosity is for the most part unimportant.

If a free surface remains sufficiently smooth and non-overhanging then the surface

can be described by a single variable giving its height above some datum. For this case Hasselmann (1971) is the only paper that to our knowledge gives a good account of the averaging process. This paper seems to have been neglected, perhaps because its main thrust was towards averaging over surface waves. It shows that there are flows of mass and momentum that need to be accounted for in the boundary conditions, as may be seen later in this paper. However, this type of analysis is inappropriate when turbulent eddies cause the interface to develop sharp cusps, overturn or become disconnected, with the formation of drops and bubbles as when splashing occurs. In this case the problem has a twofold nature as both turbulence and two-phase flow must be taken into account. That is, a situation occurs where space is occupied by two substances which have different properties (e.g. air and water) and can be distinguished from each other. The interface between the two phases is often extremely complex.

The analysis of the flow of two fluids, one dispersed throughout the other, is most often carried out by solving equations which are averaged over each single phase. This is achieved by introducing a 'phase function' or 'characteristic function of phase' for each phase, which is essentially a step function, and then averaging. The properties of the phase function are such that a number of conservation equations are obtained for each dispersed phase. Drew (1983) gives a clear and thorough review of these methods. Typical methodologies for deriving equations for two-phase flows are very similar and partly derived from to those used in the 1970s to obtain conditioned equations for single-phase turbulent/non-turbulent flows; we briefly summarize the main features of such techniques.

Conditioned equations are achieved by multiplying the flow equations by an 'intermittency function' and subsequently averaging. Corrsin & Kistler (1955) first introduced a function which analysed the sharp and irregular front which separates turbulent and non-turbulent flows. An intermittency signal $I(\boldsymbol{x}, t)$ is used to define the conditioned sampling. A theoretical analysis of intermittent turbulent flow to derive a computational intermittency model was first presented by Libby (1975). In analogy with the conditional sampling techniques used by experimentalists he used a method in which the governing equations are 'conditioned'. The method compared satisfactorily with two sets of experimental results for a two-dimensional mixing layer and for a boundary layer flow. A different formalism for conditional averaging is given by Dopazo (1977) in terms of zone averages i.e. averages defined separately for turbulent and non-turbulent flow regions. Conditioned equations are obtained in a similar fashion to Libby's but conditioned and unconditioned averages are not mixed. Chevray & Tutu (1978) successfully applied the same method to the analysis of the velocity and temperature fields of a round heated jet. Of particular relevance is their analysis of the entrainment mechanisms at the interface. Since those early works much effort has been put into the optimization of the theoretical–numerical techniques related to the intermittency theory and more recently a k–ϵ–γ turbulence model has been proposed by Cho & Chung (1992).

Application of conditional methods to two-phase flows also gave promising results. However, in the area of gas–liquid two-phase flow conditioned equations are obtained where the intermittency function is substituted by a phase function and zone averages are replaced by phase-weighted averages. As a result similar expressions are given to those obtained by means of intermittency theory for a single-phase turbulent flow (Kataoka & Serizawa 1989). Moreover, the entrainment term typically used in single-phase turbulent flow is replaced by a phase interaction term for which closure is also necessary. In both the cases of single-phase turbulent flow and of two-phase flow

the closure for this term is crucial. Among the major findings concerning turbulent two-phase flows is the observation that turbulence is one of the more important of the key parameters which determine microscopic structures in the flow (Michiyoshi & Serizawa 1984; Lahey 1987). In both solid–liquid and liquid–gas two-phase flow the presence of the dispersed phase within a connected or 'carrier' phase reduces turbulent intensities (Serizawa, Kataoka & Michiyoshi 1975). This means that two-phase turbulence is not the simple sum of the turbulence in each phase.

The similarities between the methods used for single-phase turbulent flows and two-phase flows are even further stressed in Aliod & Dopazo (1990) who developed a statistically conditioned averaging formalism for deriving two-phase flow equations. The same formalism introduced by Dopazo (1977) for single-phase turbulent flows is applied to a two-phase flow in order to overcome some conceptual and practical difficulties typical of the existing two-phase flow formalism. A k–ϵ turbulence model is used for the continuous phase and reasonably good results are found compared to experimental results obtained from a flow due to an axysimmetric jet.

A simplified version of averaging flow properties near an air–water interface by means of an intermittency function was used by Svendsen (1987) when modelling properties of a breaking water wave.

In the present work a different perspective is taken such that the air–water interactions occurring at a broken, turbulent free surface are analysed with emphasis on the liquid phase. Instead of developing governing equations for each phase we want, in the first instance, to develop appropriate boundary conditions in order to solve for the motion of the water. Except in wind-wave generation the water usually has a dominant role in the dynamics of turbulent free surfaces. Discussion of the role of air is curtailed in this paper: either neglected altogether, or given a token representation. It can and should be considered further in papers comparable with this one and Part 1.

This paper has a number of simplifications, in order to achieve greater clarity. We generally ignore any dynamic effects of the gas phase, assuming that its inertia is negligible and that drag effects may be ignored. This simplification is justified for some of the regimes discussed in Part 1, although there are also many cases where it is not justified. Similarly, we present analysis for two-dimensional motions. This is not realistic, but it is always easy to deduce the extra terms needed to extend the algebra to three dimensions, and it reduces the size of all the long and complicated equations. Also it is usually straightforward to deduce the omitted terms for fully three-dimensional flow.

In § 2 conditional and phase-weighted averaging are introduced. Section 3 is devoted to a discussion of the flow in the surface layer. Suitable surface coordinates are introduced and some of the results of conditional averaging are used to discuss the mean air–water interface. Some examples are given which refer to model air–water interfaces for different regimes. In § 4 integral equations are used to derive the basic equations in differential form for the two-phase mean flow. The equations are consistent with earlier derivations based on differential equations. In § 5 the kinematic boundary condition at the mean free surface is developed for both splashing and non-splashing regimes, followed by analysis of the dynamic boundary condition and of the boundary condition for the turbulent kinetic energy. This analysis is extended to consideration of a passive tracer in an Appendix. In § 6 the closure problem is introduced and discussed, with a simple example that builds on the classification given in Part 1. An indication of how splashing might be incorporated in closure expressions is given in § 7. The foot of spilling breakers is discussed in § 8. Since in

many cases this is the source of the turbulent structure that is the breaker, it has a special importance, especially if unsteady waves are to be modelled. It is suggested that the mean thickness and velocity of the 'surface layer', defined in § 5, may serve as instantaneous measures of the breaker strength. Section 9 gives a final discussion.

2. The conditional averaging

In this section we define averaging techniques for describing flow properties inside the two-phase 'surface layer'. Let $\langle\cdot\rangle$ denote an averaging process such that if $\mathscr{G}(\boldsymbol{x},t)$ is a generic flow variable then $\langle\mathscr{G}(\boldsymbol{x},t)\rangle = G(\boldsymbol{x},t)$ is the corresponding average. Here $\boldsymbol{x}$ is a position vector, and t is time. The most appropriate averaging process is the 'ensemble average' in which the average is evaluated at each space point as the arithmetic mean of all the single realizations $\mathscr{G}_i(\boldsymbol{x},t)$ over a set of possible 'equivalent' realizations $i = 1,\ldots N$:

$$G(\boldsymbol{x},t) = \langle\mathscr{G}(\boldsymbol{x},t)\rangle_e = \frac{1}{N}\sum_{i=1}^{N}\mathscr{G}_i(\boldsymbol{x},t). \tag{2.1}$$

Many of the flows studied in the laboratory are statistically stationary with respect to time. This means that in a suitable coordinate system velocity components at a fixed point are stationary random functions of time. If this is so, the ergodic hypothesis asserts that the mean value with respect to time

$$\langle\mathscr{G}(\boldsymbol{x},t)\rangle_t = \frac{1}{T}\int_T \mathscr{G}(\boldsymbol{x},t)\,\mathrm{d}t \tag{2.2}$$

is equivalent to the ensemble average, i.e. $\langle\mathscr{G}(\boldsymbol{x},t)\rangle_e = \langle\mathscr{G}(\boldsymbol{x},t)\rangle_t$.

The averaging process, whatever its nature, defines $G = \langle\mathscr{G}\rangle$ and satisfies

$$\langle\mathscr{F}+\mathscr{G}\rangle = \langle\mathscr{F}\rangle + \langle\mathscr{G}\rangle, \quad \langle\langle\mathscr{F}\rangle\mathscr{G}\rangle = \langle\mathscr{F}\rangle\langle\mathscr{G}\rangle, \tag{2.3}$$

$$\left\langle\frac{\partial\mathscr{G}}{\partial t}\right\rangle = \frac{\partial\langle\mathscr{G}\rangle}{\partial t}, \left\langle\frac{\partial\mathscr{G}}{\partial x_i}\right\rangle = \frac{\partial\langle\mathscr{G}\rangle}{\partial x_i}, \tag{2.4}$$

where the first two equations are called Reynolds' rules, the third is known as Leibnitz' rule and the fourth is Gauss' rule.

In order to obtain conditioned equations for each phase (i.e. air and water) we introduce a phase function or intermittency function such that for a fixed point $\boldsymbol{x}$

$$I(\boldsymbol{x},t) = \begin{cases} 1 & \text{if the point } \boldsymbol{x} \text{ is in the water at time } t \\ 0 & \text{if the point } \boldsymbol{x} \text{ is in the air at time } t. \end{cases} \tag{2.5}$$

We deal with $I(\boldsymbol{x},t)$ as a generalized function. Hence a derivative of $I(\boldsymbol{x},t)$ can be defined as a generalized function in terms of test functions which are 'sufficiently smooth' and have compact support. The intermittency factor γ is defined as follows:

$$\langle I(\boldsymbol{x},t)\rangle = \gamma(\boldsymbol{x},t). \tag{2.6}$$

It is also known as the relative residence time or volume fraction. According to equations (2.4) the intermittency factor also obeys the following:

$$\frac{\partial\gamma}{\partial t} = \frac{\partial\langle I\rangle}{\partial t} = \left\langle\frac{\partial I}{\partial t}\right\rangle, \quad \nabla\gamma = \nabla\langle I\rangle = \langle\nabla I\rangle. \tag{2.7}$$

We now define a zone average for the liquid phase (water)

$$G_w(\boldsymbol{x},t) = \langle \mathscr{G}(\boldsymbol{x},t)\rangle_w = \frac{\langle \mathscr{G}I\rangle}{\langle I\rangle} = \frac{\langle \mathscr{G}I\rangle}{\gamma} \tag{2.8}$$

and a zone average for the gas phase (air)

$$G_a(\boldsymbol{x},t) = \langle \mathscr{G}(\boldsymbol{x},t)\rangle_a = \frac{\langle \mathscr{G}(1-I)\rangle}{1-\gamma} = \frac{G-\gamma G_w}{1-\gamma}. \tag{2.9}$$

It is also clear that the conventional definition of average in a single phase can be recovered by setting $I = 1$ in equation (2.8). From these three kinds of averages, three kinds of fluctuations are distinguished by using lower-case symbols and suffices. That is

$$g = \mathscr{G} - G, \quad g_w = \mathscr{G} - G_w, \quad g_a = \mathscr{G} - G_a, \tag{2.10}$$

which all have zero mean when using the appropriate zone average, i.e. they obey

$$\langle g\rangle = 0, \quad \langle g_w\rangle_w = \langle g\rangle_w = 0, \quad \langle g_a\rangle_a = \langle g\rangle_a = 0. \tag{2.11}$$

For example to prove the above result for the liquid zone average:

$$\langle g\rangle_w = \frac{\langle(\mathscr{G} - \langle\mathscr{G}\rangle_w)I\rangle}{\gamma} = \frac{\langle \mathscr{G}I\rangle}{\gamma} - \frac{\langle\mathscr{G}\rangle_w\gamma}{\gamma} = G_w - G_w = 0. \tag{2.12}$$

Chevray & Tutu (1978) give the following relations for the means and the correlations of two flow variables $\mathscr{G}$ and $\mathscr{F}$ which link conventional and zone averages:

$$G = \langle\mathscr{G}\rangle = \gamma G_w + (1-\gamma)G_a, \tag{2.13}$$

$$\langle gf\rangle = \gamma\langle gf\rangle_w + (1-\gamma)\langle gf\rangle_a + \gamma(1-\gamma)[G_wF_w + G_aF_a - G_wF_a - G_aF_w]. \tag{2.14}$$

From Dopazo (1977) we quote an important relationship for use in the following analysis of boundary conditions:

$$\langle \mathscr{F}\mathscr{G}I\rangle = \gamma\langle\mathscr{F}\mathscr{G}\rangle_w = \gamma[\langle g_wf_w\rangle_w + F_wG_w] = \gamma[\langle gf\rangle_w + F_wG_w]. \tag{2.15}$$

Further calculation gives a relationship for the triple correlations:

$$\langle \mathscr{F}\mathscr{G}\mathscr{H}I\rangle = \gamma\langle\mathscr{F}\mathscr{G}\mathscr{H}\rangle_w = \gamma[\langle fgh\rangle_w + H_w\langle fg\rangle_w + G_w\langle fh\rangle_w + F_w\langle gh\rangle_w + F_wG_wH_w]. \tag{2.16}$$

3. The surface layer and the mean air–water interface

In this section we are concerned with the definition of the basic structure of the surface layer and lay the foundations for describing it as a particular type of two-phase flow.

We define a surface layer as the region occupied by two different phases (air and water) which, between them, fill the region. (Note the difference from some other authors who refer to the viscous boundary layer as a surface layer.) In the schematic sketch given in figure 1 the surface layer is the portion of volume enclosed by the two dashed lines which represent the lowest envelopes of bubbles and troughs in the liquid phase ('trough level') and the envelope of the highest droplets and water ejections within the gas phase ('crest level'). For practical purposes in some flow regimes, this layer may be better defined by excluding the extreme regions, populated with few drops or bubbles.

Within the surface layer flow properties such as density are not represented by continuous functions of time and space. Hence it is desirable to use mean flow

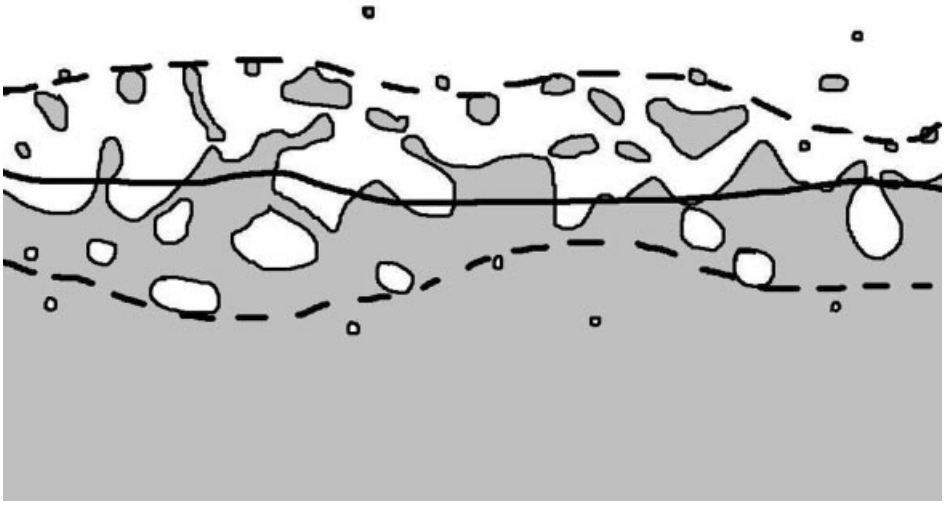

FIGURE 1. Sketch of the geometry adopted in the model for the two-phase surface layer. The upper and lower bounds of the two-phase layer are indicated with dashed lines. A possible choice of mean free surface is given with a solid line, but for our analysis a smooth lower bound of the layer is chosen.

properties, which are assumed to be continuous and smooth, to describe the dynamics of the surface layer. Here, for example, we can define the surface layer extent by using the intermittency factor γ which, at each space point, is a continuous function of time. In statistically steady flows the intermittency factor can be interpreted as the relative residence time. According to the definitions given in the previous section the trough level is such that no air is present below it, hence $\gamma(\boldsymbol{x}, t) = 1$ below trough level. Similarly no water is found above the crest level defined as $\gamma(\boldsymbol{x}, t) = 0$. In splashing turbulence more practical values such as $0.95 > \gamma > 0.05$ may be more appropriate, since the smallest drops and bubbles are convected far from the interface.

Within the surface layer let $\mathscr{S}$ be the instantaneous surface which separates the two different phases, i.e. the interface between droplets and air, between bubbles and water or between the two interpenetrating connected phases. It is common practice to describe one of the two phases as 'connected' in the sense that any two points in the specified phase can be path-connected. The other phase is consequently called 'dispersed'. The above two terms can sometimes be used without ambiguity when referring to gas–liquid two-phase flow occurring in pipes. However, the distinction between dispersed and connected phases is not so clear when describing the surface layer where no rigid boundaries are present and where the intermittency factor, or volume fraction, ranges from 0 to 1 for both phases. See the sketch in figure 1. In the top portion of the surface layer the liquid phase is not path-connected and droplets can be considered as dispersed within the air. In the central region of the surface layer both phases are path-connected. Finally in the bottom part of the surface layer the gas phase is not connected. While there is much study of flows carrying drops and of bubbly flows, there is little mention of the region where both phases are fully connected. It is given roughly by $0.3 < \gamma < 0.7$.

This description introduces in a qualitative manner the concept of mean interface $\bar{\eta}(\boldsymbol{x}, t)$ which can be assumed to be in the central portion of the surface layer where both phases are path-connected. We can immediately argue that $\bar{\eta}$ is not uniquely defined since its definition depends both on the averaging method and/or on the

statistical description used to model the dynamics of the surface layer. For a similar case see Brocchini & Peregrine's (1996) discussion of this matter for the instantaneous shoreline on a beach as it moves due to incident waves. The intermittency factor can serve to define any non-fluctuating reference level $n = n_*$ within the surface layer such that $\gamma(n_*) = \gamma_*$ with $\gamma_* = \text{const}$. An obvious choice is to take the surface $\gamma_* = 0.5$, but whilst this may be fine for some purposes, experience of averaging for water waves indicates that other choices can be more useful.

We assume that a mean interface $\bar{\eta}$, or related reference surface can be defined as a smooth time-dependent surface. This interface is then regarded as a local reference for defining the origin of a local curvilinear coordinate set $\boldsymbol{x} = (s, n, z)$. Here, s and z are local orthogonal coordinates lying in the mean surface, and n is measured normal to that surface, away from the water side. In the following the analysis is simplified by only including two-dimensional terms. This is mainly to reduce the number of terms in the long expressions that arise. Often it is easy to add the terms involving z-variation and components in the z-direction. On the other hand the curvilinear coordinates needed for (s, n, z) on a fully three-dimensional surface introduce complications that are unnecessary as we have yet to fully describe two-dimensional mean flows, where the only three-dimensional contribution comes from the turbulent fluctuations. Thus we introduce the set of unit vectors $(\hat{\boldsymbol{s}}, \hat{\boldsymbol{n}})$ as a basis for describing the motion. Here $\hat{\boldsymbol{n}}$ represents the unit vector normal to the mean free surface while $\hat{\boldsymbol{s}}$ is the unit tangent vector, which we take to be in the direction of the mean flow. The arclength s is along the mean free surface, $n = 0$. Note that the surface layer, defined as the region where two-phase flow occurs, is assumed to be thin enough that the normals do not meet inside the surface layer.

3.1. *The mean interface between air and water*

We now discuss the problem of defining the mean interface $\bar{\eta}$. To clarify some of the concepts used to define the mean air–water interface for splashing regimes we first analyse the easier conditions of a wavy, unbroken, air–water interface. In this case it is common practice to define the air–water interface (or free surface) as

$$\mathscr{S}(s, n, t) = n - \eta\,(s, t) = 0 \quad \Longrightarrow \quad n = \eta(s, t). \tag{3.1}$$

Note that this equation gives a condition for the displacement of the interface from a given reference level $n = n_*$. Hence it is essentially an equation used to refer to the interface in terms of the length η. This length is also the random variable often used in any statistical description of a continuous, turbulent air–water interface.

Let us now turn to the analysis of the air–water interface in the case of a splashing regime. For a turbulent and broken interface there is not a unique equation for $\mathscr{S}$; rather there is a collection of equations to define the boundary of disconnected portions of water. In this case the surface $\mathscr{S}$ is the union of all the surfaces $\mathscr{S}_i$ required to enclose all water portions where $\mathscr{S}_i$ bounds the ith such portion. These surfaces are labelled by

$$\mathscr{S}_i(s, n, t) = 0, \tag{3.2}$$

for each single portion of water (droplets, fingers, ejections ...).

Both Killen & Anderson (1969) and Ervine & Falvey (1987) report measurements of air–water interfaces in self-aerated flows with a conductivity probe, in a steep channel and for a turbulent jet issuing from an orifice at the downstream end of a straight pipe, respectively. Both papers imply that they measured free-surface fluctuations, in a splashing regime, though it is clear from the context and descriptions that the flow

measured is best considered as interpenetrating gas and liquid. Results show that for these air-entraining flows the free-surface fluctuations are of the same order as the total depth of the flow, and the measurements were interpreted as free-surface fluctuations around a mean value $\bar{\eta}$ approximately described by a normal distribution. Thus there is some experimental evidence that the probability density function of finding a value of η above a given n, $Pr(s,\eta)$, can be estimated with a normal distribution $\mathcal{N}(\bar{\eta},\sigma^2)$ where the variance σ is a measure of the lateral extent of the surface layer, i.e. the width of the surface layer maybe assumed to be proportional to σ.

However for splashing regimes the intermittency factor γ is the most important statistical parameter to characterize a specific surface layer. The intermittency factor provides a continuous, smooth function that can be used effectively to assess some statistical properties of a given surface layer. We can, thus, partly characterize the surface flow properties by defining the crossflow profile for γ.

For example if a Gaussian distribution is assumed for η the one-dimensional mean volume of the surface layer can be defined as the volume of the region enclosed by the boundaries $\gamma = 0$ and $\gamma = 1$ where the intermittency factor γ is now defined according to

$$\gamma(s,n,t) = \frac{1}{2}\left[1 - \mathrm{erf}\left(\frac{n-\bar{\eta}(s,t)}{\sqrt{2}\sigma(s,t)}\right)\right]. \tag{3.3}$$

Clearly the intermittency factor only asymptotically reaches zero (i.e. for $n \to \infty$). However, at $n = 2\sigma$ it has the value 0.05 and is effectively zero ($\gamma \approx 10^{-3}$) for $n = 3\sigma$ which is the normalization chosen in figure 3.

3.2. *The intermittency factor for different air–water interfaces*

Since we believe that the intermittency factor γ is an important statistical parameter for characterizing a surface layer we compute three representative examples of γ for different surface layers of width normalized to $(-1,1)$ which might be used to represent three different flow regimes:

(*a*) a wavy, smooth air–water interface is given by

$$\eta = n = \cos(\omega t) \tag{3.4}$$

and shown in figure 2. The intermittency factor γ is simply obtained as

$$\gamma(n) = \frac{1}{\pi}\cos^{-1}(n) \tag{3.5}$$

and the profile of γ is shown in figure 3 with a dashed line.

(*b*) For a regime in which the turbulent flow is restrained by gravity but is large enough to generate 'scars' at the free surface we use a simple model to represent periodic scars of period T (see figure 2). We choose to describe the free surface with a power law:

$$\eta = n = 1 - 2\left(\frac{t}{T}\right)^m, \quad m \in \mathbb{N} \quad \text{and even.} \tag{3.6}$$

Consequently γ can be computed as

$$\gamma(n) = \frac{t(n)}{T} = \left(\frac{1-n}{2}\right)^{1/m}, \quad m \in \mathbb{N} \quad \text{and even.} \tag{3.7}$$

and this profile is shown in figure 3 with a dot-dashed line for $m = 4$.

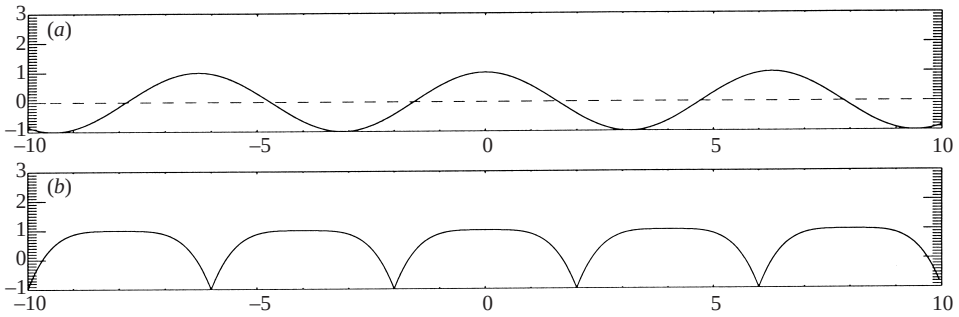

FIGURE 2. Free surface profile for: (*a*) a sinusoidal wavy interface, (*b*) a periodic scarified interface.

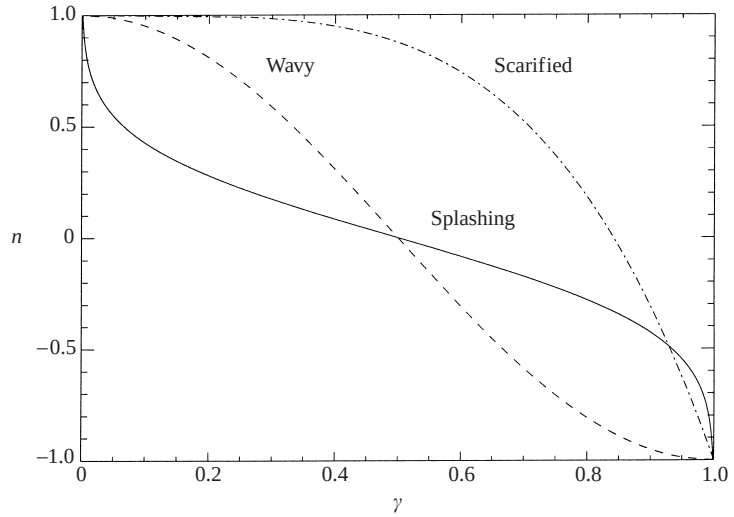

FIGURE 3. The intermittency factor, γ, for: a wavy air–water interface (dashed line), a periodically scarified interface (dot-dashed line) as from equation (3.7) with $m = 4$ and a turbulent splashing interface (solid line).

(*c*) Finally, for a turbulent splashing regime we assume the intermittency factor associated with a normal distribution $\mathcal{N}(\bar{\eta}, \sigma^2)$ (see equation (3.3)). For this case the profile through the surface layer is displayed in figure 3 with a solid line using a cut-off of 0.001.

Comparison of the three γ-factors above shows how the surface layer can vary for different flow regimes. In particular it is evident that for wavy regimes water is present at the top of the surface layer ($n = 1$) for a longer time than for a splashing regime in which water drops are present for a shorter time near the top of the layer. This behaviour is reversed at the bottom of the layer where values of γ near 1 are reached faster while approaching the base of the layer ($n = -1$) in a splashing regime. The intermittency factor is nearly 1 for most of a periodic scarified air–water interface and γ asymptotically tends to 1 for the whole surface layer thickness as m increases. Similar results would be found if vortex dimples and vortex cores were included in the surface deformations.

4. Integral and differential equations of motion

In this section we derive the basic conservation equations for the flow within the surface layer. While it is clear that conservation equations for mass and linear momentum are required, the number of transport equations for the turbulent properties (e.g. turbulent kinetic energy, Reynolds' stresses, etc.) depends on the chosen level of closure. We derive the primary mass and momentum conservation equations together with a transport equation for the turbulent kinetic energy k.

There are several approaches for deriving equations for the motion of a mixture. The simplest is the 'phenomenological approach' in which conservation laws are written for each of the two phases separately with additional terms in each to model the interaction of the two phases. The main disadvantage of this intuitive approach is that it usually involves the *ad-hoc* introduction of the additional terms for the phase interaction. A more reliable method is the 'averaging method' in which equations for the macroscopic behaviour of the flow are obtained by averaging equations which are valid at a microscopic level. In this class two different approaches coexist. One is based on the assumption that each phase involved in the mixture can be described as a continuum governed by conservation laws which are expressed in differential form. These equations are obtained by 'conditioning' the conservation equations for a single-phase flow (e.g. Dopazo 1977 and Drew 1983). We do not believe that the main assumption of dealing with each phase as a continuum (and the consequent use of differential equations) is in principle correct when we are averaging over multiple disconnected regions. Hence, we use a second approach which is based of the conditioning of integral equations which are valid for a control volume of the mixture. This method can be seen as a revised and simplified (more straightforward definition of the averaging process) version of the work by Drew (1971).

We, thus, introduce a fixed control volume $\mathbb{V}$ bounded by a surface $\mathbb{S}$ in a region where both phases are present. Figure 4 illustrates a control volume. The photograph shows the region of disconnected water and in the lower portion of $\mathbb{V}$ the region where both phases are connected. In the most general case where change of phase occurs the surface of discontinuity $\mathscr{S}$ separating the two different phases is not a material interface, i.e. the velocity of an interface does not coincide with the velocity of the medium: $\mathscr{U}_{s_i} \neq \mathscr{U}_i$. In principle in the generic case of gas–liquid two-phase flow the gas can be absorbed by the liquid and diffuse through it, and there may be change of phase as in evaporation, boiling or cavitation.

In order to simplify our task of defining appropriate boundary conditions at a water surface disrupted by strong turbulence we take advantage of the density difference between the two fluids (air and water) characterizing the specific two-phase flow problem of a turbulent, splashing air–water interface. The large density and viscosity ratios between air and water permit the introduction of a simplification of zone averaging by neglecting dynamical effects due to the air motion. Although this simplified approach can only be used for describing flow regimes in which air flow has negligible effect, this is a fairly wide class of flows and it clarifies the following analysis. Even so, in the following analysis we often indicate such interaction terms.

4.1. *The conservation of mass*

The balance of mass requires that the rate of change of the total mass in the volume $\mathbb{V}$ is balanced by the inflow of mass across the boundary $\mathbb{S}$. There is clearly a jump condition across interfaces $\mathscr{S}$ because they need not be material surfaces, hence we

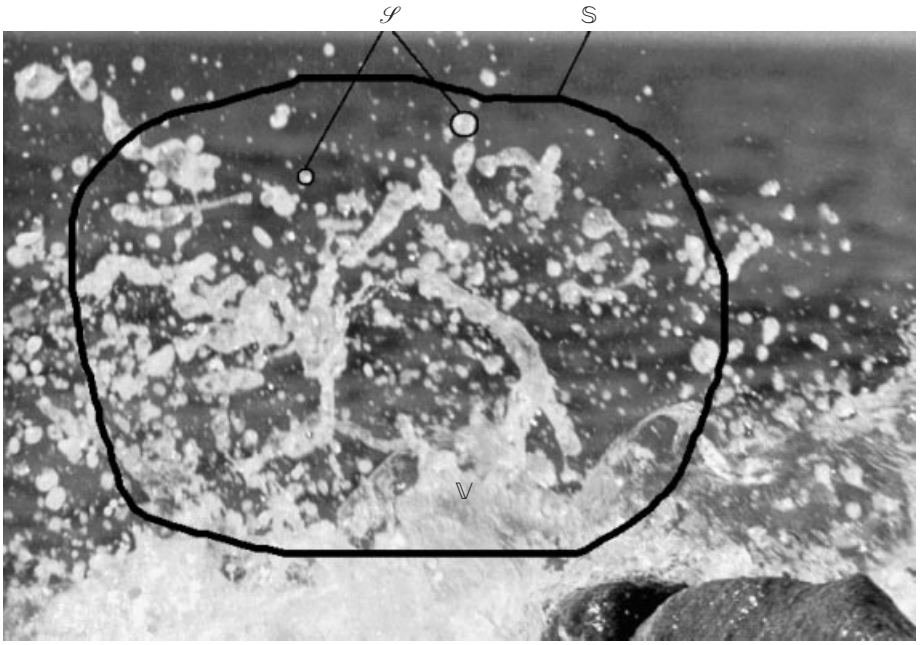

FIGURE 4. The air–water continuum. The thick closed curve represents the boundary $\mathbb{S}$ of the fixed control volume $\mathbb{V}$. An example of the discontinuity interface $\mathscr{S}$ between the two phases is drawn around two water droplets. This photograph has a width of about 20 cm.

have, for the liquid phase only:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{V}}\langle\rho I\rangle\,\mathrm{d}\mathbb{V}+\int_{\mathbb{S}}\langle\rho\mathscr{U}_i I\rangle e_i\mathrm{d}\mathbb{S}=-\left\langle\int_{\mathscr{S}}\rho(\mathscr{U}_i-\mathscr{U}_{s_i})\tilde{n}_i\mathrm{d}\mathscr{S}\right\rangle \tag{4.1}$$

where $\tilde{n}$ is the normal exterior to the liquid phase and $\boldsymbol{e}$ is the normal exterior to the surface $\mathbb{S}$ of the control volume.

The surface contribution is such that the ensemble average must be carried out outside the integrals in order to consider all contributions due to the motion of the surface $\mathscr{S}$. In the following we assume that integration is over a fixed control volume $\mathbb{V}$, then averaging commutes:

$$\left\langle\int_{\mathbb{V}}\mathscr{F}\mathrm{d}\mathbb{V}\right\rangle=\int_{\mathbb{V}}\langle\mathscr{F}\rangle\,\mathrm{d}\mathbb{V},\quad\text{but}\quad\left\langle\int_{\mathscr{S}}\mathscr{F}\tilde{\boldsymbol{n}}\mathrm{d}\mathscr{S}\right\rangle\neq\int_{\mathscr{S}}\langle\mathscr{F}\rangle\tilde{\boldsymbol{n}}\mathrm{d}\mathscr{S}. \tag{4.2}$$

We also use a second result for the gradient of the intermittency function. This can be defined following the theory of generalized functions. If $\mathscr{F}$ is a smooth function except at the interface $\mathscr{S}$ then:

$$\left\langle\int_{\mathscr{S}}\mathscr{F}\tilde{\boldsymbol{n}}\mathrm{d}\mathscr{S}\right\rangle=-\int_{\mathbb{V}}\langle\mathscr{F}\nabla I\rangle\mathrm{d}\mathbb{V}\quad\text{or}\quad\left\langle\int_{\mathscr{S}}\mathscr{F}\tilde{n}_i\mathrm{d}\mathscr{S}\right\rangle=-\int_{\mathbb{V}}\left\langle\mathscr{F}\frac{\partial I}{\partial x_i}\right\rangle\mathrm{d}\mathbb{V}. \tag{4.3}$$

We define I as a generalized function to relate the integral over a control volume for the air–water mixture into an integral over the air–water interface. Drew (1983) proves by means of suitable test functions that ∇I behaves like a delta function picking out the interface $\mathscr{S}$. This is not surprising as the intermittency function I is a Heaviside-type function (step-function) and hence $\nabla I(\boldsymbol{x},t)\propto\delta(\mathbf{x},t)$ where δ is Dirac's

delta function. Further ∇I is directed towards the interior of the water phase, i.e.

$$\nabla I(\boldsymbol{x}, t) = -\tilde{\boldsymbol{n}}\delta(\boldsymbol{x}, t) \quad \boldsymbol{x} \in \mathscr{S}. \tag{4.4}$$

This directly gives

$$\int_{\mathbb{V}} \mathscr{F} \nabla I \, \mathrm{d}\mathbb{V} = -\int_{\mathscr{S}} \mathscr{F} \tilde{\boldsymbol{n}} \mathrm{d}\mathscr{S}. \tag{4.5}$$

Applying the above results to the equation of conservation of mass we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{V}} \langle \rho I \rangle \mathrm{d}\mathbb{V} + \int_{\mathbb{S}} \langle \rho \mathscr{U}_i I \rangle \mathrm{d}\mathbb{S} = \int_{\mathbb{V}} \Delta \mathrm{d}\mathbb{V}, \quad \text{where} \quad \Delta = \left\langle \rho(\mathscr{U}_i - \mathscr{U}_{s_i}) \frac{\partial I}{\partial x_i} \right\rangle \tag{4.6}$$

and the term on the right-hand side represents the average supply of mass to the liquid phase due to phase changes. Applying the divergence theorem gives

$$\int_{\mathbb{V}} \left[\frac{\partial \langle \rho I \rangle}{\partial t} + \frac{\partial \langle \rho \mathscr{U}_i I \rangle}{\partial x_i} - \Delta \right] \mathrm{d}\mathbb{V} = 0. \tag{4.7}$$

Since the control volume is arbitrary the above equation can be rearranged to obtain a differential equation for the scalar γ with the assumption of constant density in the liquid phase. After using (2.7) the equation for γ becomes

$$\frac{\partial \gamma}{\partial t} + \frac{\partial \langle \mathscr{U}_i I \rangle}{\partial x_i} = \frac{\partial \gamma}{\partial t} + \frac{\partial (\gamma U_{wi})}{\partial s_i} = \Delta \tag{4.8}$$

where Δ can be interpreted as a phase interaction term (see for instance Aliod & Dopazo 1990). A similar equation holds for the mass conservation in a turbulent/non-turbulent single-phase flow (Dopazo 1977).

For our analysis of the turbulent air–water interface under the assumption that no change of phase occurs (i.e. $\mathscr{U}_{s_i} = \mathscr{U}_i$) the source term Δ is zero. In general it would be non-zero for cavitating, condensing or evaporating liquid and for gases in solution. In the following applications, however, we assume that no change of phase occurs and we neglect all such phase-interaction contributions in the governing equations. This reasonable assumption saves us finding a closure for Δ and the transport equation for γ is

$$\frac{\partial \gamma}{\partial t} + \frac{\partial (\gamma U_{wi})}{\partial x_i} = 0. \tag{4.9}$$

4.2. *The conservation of linear momentum*

We now analyse the conservation of linear momentum using the same approach as for the equation of conservation of mass. If no phase interaction term Δ occurs we require that the rate of change of linear momentum within the control volume equals the sum of four contributions, namely the flux of momentum across the boundary $\mathbb{S}$, the external gravity force $\rho \boldsymbol{g}$, the total internal forces exerted on the mass within the volume $\mathbb{V}$ through its bounding surface and the interfacial forces due to the stress tensor $\mathscr{T}$ acting on the interface $\mathscr{S}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{V}} \langle \rho \mathscr{U}_i I \rangle \mathrm{d}\mathbb{V} + \int_{\mathbb{S}} \langle \rho \mathscr{U}_i \mathscr{U}_j I \rangle e_j \mathrm{d}\mathbb{S} = \int_{\mathbb{V}} \langle \rho g_i I \rangle \mathrm{d}\mathbb{V} + \int_{\mathbb{S}} \langle \mathscr{T}_{ij} I \rangle e_j \mathrm{d}\mathbb{S} + \left\langle \int_{\mathscr{S}} \mathscr{T}_{ij} \tilde{n}_j \mathrm{d}\mathscr{S} \right\rangle. \tag{4.10}$$

In particular we can rewrite the last contribution by using (4.3):

$$\left\langle \int_{\mathscr{S}} \mathscr{T}_{ij} \tilde{n}_j \mathrm{d}\mathscr{S} \right\rangle = -\int_{\mathbb{V}} \left\langle \mathscr{T}_{ij} \frac{\partial I}{\partial x_j} \right\rangle \mathrm{d}\mathbb{V}. \tag{4.11}$$

This result is substituted back into the conservation equation which, after applying Green's theorem, becomes

$$\int_{\mathbb{V}} \left[\frac{\partial \langle \rho \mathscr{U}_i I \rangle}{\partial t} + \frac{\partial \langle \rho \mathscr{U}_i \mathscr{U}_j I \rangle}{\partial x_j} - \langle \rho g_i I \rangle - \frac{\partial \langle \mathscr{T}_{ij} I \rangle}{\partial x_j} + \left\langle \mathscr{T}_{ij} \frac{\partial I}{\partial x_j} \right\rangle \right] \mathrm{d}\mathbb{V} = 0 \qquad (4.12)$$

giving the required differential equation for the mean water flow ($\rho = \text{const.}$):

$$\frac{\partial \langle \mathscr{U}_i I \rangle}{\partial t} + \frac{\partial \langle \mathscr{U}_i \mathscr{U}_j I \rangle}{\partial x_j} - g_i \langle I \rangle - \frac{1}{\rho} \frac{\partial \langle \mathscr{T}_{ij} I \rangle}{\partial x_j} + \frac{1}{\rho} \left\langle \mathscr{T}_{ij} \frac{\partial I}{\partial x_j} \right\rangle = 0. \qquad (4.13)$$

The last term, representing the interfacial momentum transfer, depends on $\partial I / \partial x_j$ which is a delta-function in the direction normal to the interface between air and water (see the previous section). We refer to Ishii (1975) for a detailed analysis of this term. In general it needs proper and specific modelling, i.e. constitutive equations are required to represent the interfacial forces. Among the most relevant models are those by Drew & Lahey (1979), Drew (1983) and Drew & Lahey (1987).

Since water is an incompressible Newtonian fluid to a good approximation we expand the terms containing the stress tensor $\mathscr{T}$ using the following constitutive equation:

$$\mathscr{T}_{ij} = -\mathscr{P} \delta_{ij} + \nu \rho \mathscr{E}_{ij} \quad \text{with} \quad \mathscr{E}_{ij} = \left(\frac{\partial \mathscr{U}_i}{\partial x_j} + \frac{\partial \mathscr{U}_j}{\partial x_i} \right), \qquad (4.14)$$

where $\nu \rho \mathscr{E}_{ij}$ is the viscous stress tensor and ν is the kinematic viscosity.

After some manipulation we obtain for the conservation of linear momentum

$$\frac{\partial \langle \mathscr{U}_i I \rangle}{\partial t} + \frac{\partial \langle \mathscr{U}_i \mathscr{U}_j I \rangle}{\partial x_j} - g_i \langle I \rangle + \frac{1}{\rho} \frac{\partial \langle \mathscr{P} I \rangle}{\partial x_i} - \nu \frac{\partial}{\partial x_j} \left(\frac{\partial \langle \mathscr{U}_i I \rangle}{\partial x_j} + \frac{\partial \langle \mathscr{U}_j I \rangle}{\partial x_i} \right) + \frac{\Phi_i}{\rho} + \Xi_i = 0. \qquad (4.15)$$

where Φ_i and Ξ_i are respectively the inviscid and viscous contributions to the interfacial momentum transfer:

$$\Phi_i = -\left\langle \mathscr{P} \frac{\partial I}{\partial x_i} \right\rangle, \qquad (4.16)$$

$$\Xi_i = \nu \left\langle \mathscr{E}_{ij} \frac{\partial I}{\partial x_j} \right\rangle + \nu \frac{\partial}{\partial x_j} \left(\left\langle \mathscr{U}_i \frac{\partial I}{\partial x_i} \right\rangle + \left\langle \mathscr{U}_j \frac{\partial I}{\partial x_i} \right\rangle \right). \qquad (4.17)$$

Here we consider high Reynolds number flows and thus neglect viscous stresses within the water, which are generally much less effective than turbulent stresses in transporting momentum. We also neglect the viscous interfacial momentum transfer although we note that its effects on the surfaces of drops and bubbles can be substantial. Consequently the equation for the conservation of linear momentum simplifies to

$$\frac{\partial \langle \mathscr{U}_i I \rangle}{\partial t} + \frac{\partial \langle \mathscr{U}_i \mathscr{U}_j I \rangle}{\partial x_j} - g_i \langle I \rangle + \frac{1}{\rho} \frac{\partial \langle \mathscr{P} I \rangle}{\partial x_i} + \frac{\Phi_i}{\rho} = 0. \qquad (4.18)$$

Note that the surface tension τ which represents a discontinuity of the pressures across the interface is here incorporated in the interfacial forces Φ_i experienced by the water.

4.3. *The balance equation for the turbulent kinetic energy*

We now want to define a conservation equation for the turbulent kinetic energy of the water:

$$k = \tfrac{1}{2}\langle u_i u_i \rangle_w = \tfrac{1}{2}\langle q^2 \rangle_w \quad \text{where} \quad q^2 = u_i u_i. \tag{4.19}$$

In order to obtain a model equation for k it is common practice to use the differential momentum equation for the instantaneous flow variables. Because of our assumptions that model equations in differential form are only valid for mean flow properties we start from integral equations for the mean mechanical energy of the system and for the mean linear momentum. While for the analysis of momentum transport inside the surface layer it is sufficient to consider the action of the large eddies (of size δ) and viscosity effects can be neglected (see § 4.2) this is not true when defining the transport equation for the turbulent kinetic energy. It is known that dissipation of turbulent kinetic energy occurs by viscosity at a much smaller length scale, i.e. at the Kolmogorov length scale in a single fluid, but at the dimensions of drops and bubbles in a two-phase flow. The generation of small-scale fluctuations is due to the nonlinear terms in the equation of motion; the viscous terms prevent the generation of infinitely small scales of motion by dissipating small-scale energy into heat (Tennekes & Lumley 1972). Hence viscous terms must be retained when deriving the balance equation for k. Assuming no contributions due to change of phase the equation for the mean mechanical energy is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{V}} \frac{\langle \rho \mathscr{U}_i^2 I\rangle}{2}\mathrm{d}\mathbb{V} + \int_{\mathbb{S}} \frac{\langle \rho \mathscr{U}_i^2 \mathscr{U}_j I\rangle}{2} e_j \mathrm{d}\mathbb{S} &= \int_{\mathbb{V}} \langle \rho g_i \mathscr{U}_i I\rangle \mathrm{d}\mathbb{V} + \int_{\mathbb{S}} \langle \mathscr{T}_{ij}\mathscr{U}_i I\rangle e_j \mathrm{d}\mathbb{S} \\ &\quad - \int_{\mathbb{V}} \left\langle \mathscr{T}_{ij}\frac{\partial \mathscr{U}_i}{\partial x_j} I \right\rangle \mathrm{d}\mathbb{V} + \left\langle \int_{\mathscr{S}} \mathscr{T}_{ij}\mathscr{U}_i \tilde{n}_j \mathrm{d}\mathscr{S} \right\rangle.\end{aligned} \tag{4.20}$$

In words, the rate of change of the mean kinetic energy inside the fixed control volume $\mathbb{V}$ is the sum of five contributions. They are the flux of mean kinetic energy across the boundary $\mathbb{S}$, the rate at which gravity does work, the rate at which the surface stresses do work, the rate at which internal stresses do work and the rate at which stresses at the air–water interface do work.

Following the procedure used for the mass and momentum equations this integral equation reduces to a differential equation for the mean mechanical energy:

$$\frac{1}{2}\frac{\partial \langle \mathscr{U}_i^2 I\rangle}{\partial t} + \frac{1}{2}\frac{\partial \langle \mathscr{U}_i^2 \mathscr{U}_j I\rangle}{\partial x_j} - \langle g_i \mathscr{U}_i I\rangle - \frac{1}{\rho}\frac{\partial \langle \mathscr{T}_{ij}\mathscr{U}_i I\rangle}{\partial x_j} + \frac{1}{\rho}\left\langle \mathscr{T}_{ij}\frac{\partial \mathscr{U}_i}{\partial x_j} I\right\rangle + \frac{1}{\rho}\left\langle \mathscr{T}_{ij}\mathscr{U}_i \frac{\partial I}{\partial x_j}\right\rangle = 0. \tag{4.21}$$

In particular the contribution due to the rate of working of internal stresses can be simplified by using the constitutive equation (4.14) and assuming incompressibility of the water:

$$\begin{aligned}\frac{1}{\rho}\left\langle \mathscr{T}_{ij}\frac{\partial \mathscr{U}_i}{\partial x_j} I\right\rangle &= \frac{1}{\rho}\left\langle (-\mathscr{P}\delta_{ij} + \nu\rho\mathscr{E}_{ij})\frac{\partial \mathscr{U}_i}{\partial x_j} I\right\rangle \\ &= \frac{1}{\rho}\left\langle \left(-\mathscr{P}\frac{\partial \mathscr{U}_i}{\partial x_j} + \frac{\nu\rho}{2}\mathscr{E}_{ij}\mathscr{E}_{ij}\right) I\right\rangle = \frac{\nu}{2}\langle \mathscr{E}_{ij}\mathscr{E}_{ij} I\rangle.\end{aligned} \tag{4.22}$$

We now subtract from the equation for the balance of mechanical energy the scalar

product of the equation for the mean linear momentum with the mean water velocity U_{w_i}. After some tedious and complicated algebra which involves the use of equations (2.15) and (2.16) we obtain the required equation:

$$\frac{\partial(\gamma k)}{\partial t}+\frac{\partial(\gamma k U_{w_j})}{\partial x_j}+\gamma\langle u_i u_j\rangle_w\frac{\partial(U_{w_i})}{\partial x_j}+\frac{\partial}{\partial x_j}[\gamma\langle u_j k\rangle_w]+\frac{1}{\rho}\frac{\partial\langle p u_j\rangle_w}{\partial x_j}-\nu\frac{\partial}{\partial x_j}\left[\frac{\partial(\gamma k)}{\partial x_j}\right]$$

$$-\nu\frac{\partial}{\partial x_j}\left[\frac{\partial(\gamma\langle u_i u_j\rangle_w)}{\partial x_i}\right]+\gamma\epsilon+\frac{\Psi}{\rho}+\Upsilon+\frac{\gamma P_w}{\rho}\frac{\partial U_{w_j}}{\partial x_j}-\nu\gamma\left[\left(\frac{\partial U_{wj}}{\partial x_j}\right)^2+U_{w_j}\frac{\partial^2 U_{wi}}{\partial x_i\partial x_j}\right]=0 \tag{4.23}$$

where

$$\Psi=-\left\langle\mathscr{P}\mathscr{U}_j\frac{\partial I}{\partial x_j}\right\rangle+U_{w_j}\left\langle\mathscr{P}\frac{\partial I}{\partial x_j}\right\rangle=-\left\langle\mathscr{P}(\mathscr{U}_j-U_{w_j})\frac{\partial I}{\partial x_j}\right\rangle=-\left\langle\mathscr{P}u_{w_j}\frac{\partial I}{\partial x_j}\right\rangle, \tag{4.24}$$

$$\Upsilon=\nu\frac{\partial}{\partial x_j}\left[\left\langle\left(\frac{\mathscr{U}_i^2}{2}+\mathscr{U}_i\mathscr{U}_j\right)\frac{\partial I}{\partial x_j}\right\rangle\right]-\nu U_{w_i}\frac{\partial}{\partial x_j}\left[\left\langle\mathscr{U}_i\frac{\partial I}{\partial x_j}\right\rangle+\left\langle\mathscr{U}_j\frac{\partial I}{\partial x_i}\right\rangle\right]$$
$$+\nu\left[\left\langle\mathscr{E}_{ij}(\mathscr{U}_j-U_{w_j})\frac{\partial I}{\partial x_j}\right\rangle\right], \tag{4.25}$$

$$\epsilon=\frac{\nu}{2}\langle\epsilon_{ij}\epsilon_{ij}\rangle_w=\frac{\nu}{2}\left\langle\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right)\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right)\right\rangle_w. \tag{4.26}$$

The first two terms in equation (4.23) can be interpreted as showing that k is conserved except for its production and dissipation by the remaining terms. The third term represents the production of turbulent kinetic energy by the mean rate of strain (or equivalently the extraction of mean flow energy by the turbulence). The fourth term is the diffusion due to turbulent velocity fluctuations while the fifth is the working of pressure fluctuations. The sixth and the seventh represent the viscous diffusion of turbulent kinetic energy. The eight term, $\gamma\epsilon$, is the viscous dissipation of turbulent kinetic energy. The ninth term, Ψ, represents the inviscid interfacial transport of turbulent energy which can only be made explicit after closure arguments are introduced. The tenth, Υ, is the interfacial transport of turbulent kinetic energy due to viscosity.

Finally there are three extra work terms due to the mean pressure and to the mean rate of strain. They arise because of the particular method we use to obtain the equation for k: using an integral method introduces these extra terms which are not present in equations obtained through a differential formulation of the conservation laws. Analysis of these contributions immediately shows that they only arise in the two-phase region. Below trough level ($\gamma=1$) the fluid is incompressible hence $\partial U_{w_j}/\partial x_j=\partial U_j/\partial x_j=0$ while above the layer $\gamma=0$. In particular, manipulation of the work term due to the mean pressure and use of equation (4.9) shows that

$$\frac{\gamma P_w}{\rho}\frac{\partial U_{w_j}}{\partial x_j}=-\frac{P_w}{\rho}\frac{\mathrm{D}\gamma}{\mathrm{D}t}, \tag{4.27}$$

which is very similar to the rate of work done by pressure in case of a compressible

fluid

$$\frac{P}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}. \tag{4.28}$$

In this case the intermittent presence of water makes the surface layer behave like a compressible single-phase fluid.

A derivation of the corresponding equations for the convection and diffusion of a passive tracer is in the Appendix.

5. The boundary conditions

In this section we derive the appropriate averaged boundary conditions to be applied either at a mean air–water interface $n = \bar{\eta}$ or other mean surface such as the base of the surface layer $n = b$. In either case the assumption of quasi-two-dimensional flow holds as described in § 3 and we introduce a pair of inertial local Cartesian coordinates (s, n), such that $\hat{\boldsymbol{s}}$ is roughly parallel to the 'surface'. Curvilinear, non-inertial, coordinates include further details which obscure the main derivation and follow in Part 3 (Brocchini & Peregrine 2002).

Although we may define the mean free surface $\bar{\eta}$ for the air–water mixture in terms of the integral of the intermittency factor we find that it is more appropriate to define the boundary conditions at the base of the surface layer. This follows Brocchini & Peregrine (1996) where the boundary conditions for the swash zone flows have been most conveniently obtained at the seaward boundary of the swash zone rather than at any mean shoreline.

General expressions for use at turbulent, discontinuous interfaces are obtained by integrating across the two-phase layer and some comparison is made when possible (e.g. kinematic boundary condition) with the boundary conditions valid for a continuous interface. This integral approach implies that the two-phase surface layer is relatively thin compared with the rest of the flow, and that the surfaces defined by $n =$ constant throughout the layer are sufficiently smooth. Thus strongly splashing regimes cannot be described in this way, since in such cases the two-phase regions are often large.

5.1. *The kinematic boundary condition*

If the 'surface layer' is taken to be represented by a smooth surface then modelling its motion involves the kinematic boundary condition for the liquid phase:

$$\frac{\partial \eta}{\partial t} + \mathscr{U}|_{\eta}\frac{\partial \eta}{\partial s} - \mathscr{V}|_{\eta} = 0, \tag{5.1}$$

where $(\mathscr{U}, \mathscr{V})$ are the streamwise and normal components of the velocity field and η is the coordinate of the air–water interface. This differential equation can only be applied under the assumption of a free surface with gentle slopes, i.e. for weak turbulence. The kinematic boundary condition states that particles which constitute the air–water interface will always remain on the interface, but this is not the case for a broken, or scarred, interface.

We first use the above results to derive the kinematic boundary condition at the mean free-surface for flow regimes where no free-surface breakup occurs. The boundary conditions obtained are thus only valid for the wavy, rippled and knobbly regimes. However, a different method is used to obtain the kinematic boundary condition for a flow in the splashing regime. In this case the free surface is no longer continuous and methods which require regularity of the flow variables near the

surface cannot be used. The kinematic boundary condition for a continuous interface is discussed first to illustrate similarities to, and differences from, the corresponding result obtained by two-phase flow arguments. Basic conservation laws for the mass and momentum flow inside the 'surface layer' are used to develop the boundary conditions for the disconnected regimes.

In the analysis of non-splashing regimes we follow the method given by Hasselmann (1971). For the instantaneous flow a kinematic boundary condition can be defined such that no water particle can leave the free surface itself. This is not true for the mean flow and an extra normal-to-surface velocity W must be supplied to model the outflow or inflow across the mean surface. The problem of assessing the boundary conditions for the mean flow can be reduced to defining a suitable transfer velocity W.

The method described by Hasselmann (1971) adopts a conventional average and consists of three steps which in our terminology are:

(*a*) define the kinematic boundary condition for the mean flow. This is achieved by writing the boundary condition in the style of (5.1) at the mean free surface $n = \bar{\eta}$:

$$\frac{\partial \bar{\eta}}{\partial t} + \mathscr{U}|_{\bar{\eta}} \frac{\partial \bar{\eta}}{\partial s} - \mathscr{V}|_{\bar{\eta}} = \mathscr{W}|_{\bar{\eta}}, \tag{5.2}$$

and by applying an ordinary average such that

$$\frac{\partial \bar{\eta}}{\partial t} + U|_{\bar{\eta}} \frac{\partial \bar{\eta}}{\partial s} - V|_{\bar{\eta}} = W|_{\bar{\eta}}; \tag{5.3}$$

(*b*) average the kinematic boundary condition for the instantaneous flow

$$\left\langle \frac{\partial \eta}{\partial t} + \mathscr{U}|_{\bar{\eta}} \frac{\partial \eta}{\partial s} - \mathscr{V}|_{\bar{\eta}} \right\rangle = 0; \tag{5.4}$$

(*c*) evaluate W by subtracting (5.3) from (5.4). Then insert W into (5.3) and after some algebra and use of the continuity equation for the instantaneous flow we get

$$\frac{\partial \bar{\eta}}{\partial t} + U|_{\bar{\eta}} \frac{\partial \bar{\eta}}{\partial s} - V|_{\bar{\eta}} = -\left\langle \frac{\partial}{\partial s} \int_{\bar{\eta}}^{\eta} \mathscr{U} \, \mathrm{d}n \right\rangle, \tag{5.5}$$

where a closure is needed for the surface flow term on the right-hand side. Note that when the free surface η is below the mean level $\bar{\eta}$ the velocity field is implicitly assumed to be extended analytically up to the mean level (Longuet-Higgins & Stewart 1964).

A formulation of the kinematic boundary condition for the mean flow for disconnected surfaces can be given which does not rely on the smoothness characteristics of the free surface η but on the behaviour of the intermittency factor γ. Since we are interested in the boundary condition at a non-fluctuating level we choose to integrate the evolution equation for the intermittency factor (4.9) over the layer defined by the normal coordinates $n = b(s,t)$ and $n = h(s,t)$, the latter being a non-fluctuating reference level coinciding with the level of an envelope of the highest splashes and crests (i.e. $\gamma|_h = 0$) and the former the base of the layer of entrained bubbles and lowest troughs (i.e. $\gamma|_b = 1$):

$$\int_b^h \left[\frac{\partial \gamma}{\partial t} + \frac{\partial (\gamma U_{w_i})}{\partial x_i} \right] \mathrm{d}n = 0. \tag{5.6}$$

Applying Leibnitz' rule for differentiating an integral function gives

$$\frac{\partial}{\partial t}\int_b^h \gamma\,\mathrm{d}n - \gamma|_h\frac{\partial h}{\partial t} + \gamma|_b\frac{\partial b}{\partial t} + \frac{\partial}{\partial s}\int_b^h \gamma U_w\,\mathrm{d}n$$

$$-(\gamma U_w)|_h\frac{\partial h}{\partial s} + (\gamma U_w)|_b\frac{\partial b}{\partial s} + (\gamma V_w)|_h - (\gamma V_w)|_b = 0. \quad (5.7)$$

This is simplified by noting that $\gamma|_h = 0$, $\gamma|_b = 1$ and $U_{w_i} = U_i$ to give

$$\frac{\partial}{\partial t}\int_b^h \gamma\mathrm{d}n + \frac{\partial b}{\partial t} + \frac{\partial}{\partial s}\int_b^h \gamma U_w\mathrm{d}n + U|_b\frac{\partial b}{\partial s} - V|_b = 0. \quad (5.8)$$

We now recognize that the integral in the first term is the amount of water above the base of the layer, hence we define

$$d(s,t) \equiv \int_b^h \gamma\mathrm{d}n. \quad (5.9)$$

This gives a further natural choice for the mean air–water interface as follows:

$$\bar{\eta} = b + d \equiv b + \int_b^h \gamma\mathrm{d}n. \quad (5.10)$$

With these definitions we can rewrite the kinematic boundary condition in a more familiar way:

$$\frac{\partial b}{\partial t} + U|_b\frac{\partial b}{\partial s} - V|_b = -W = -\left(\frac{\partial d}{\partial t} + \frac{\partial}{\partial s}\int_b^h \gamma U_w\mathrm{d}n\right). \quad (5.11)$$

On the right hand side of this equation we find terms describing the conservation of mass (d) inside the surface layer. Hence the equation can be regarded either as the kinematic boundary condition for the flow below the base of the layer or as an equation for the conservation of mass inside the surface layer. The mass in the layer is increased/decreased due to inflow/outflow $V|_b$ in a time δt, by the inflow/outflow $U|_b\partial b/\partial s$ due to the slope of the base level and to the inflow/outflow due to the rate of change $\partial b/\partial t$ of the base level. This equation is formally very similar to (5.5) but it is applied at the base of the layer rather than at the mean interface $\bar{\eta}$. For this reason the extra term $\partial d/\partial t$ representing the rate of change of the water volume inside the layer is present.

By employing equations (5.9) and (5.10) we can write the kinematic boundary condition more compactly as

$$\frac{\partial\bar{\eta}}{\partial t} + U|_b\frac{\partial b}{\partial s} - V|_b = -\frac{\partial}{\partial s}\int_b^h \gamma U_w\mathrm{d}n. \quad (5.12)$$

This equation is formally similar to (5.5) and we also recover the surface flow type contribution. In both (5.11) and (5.12) this contribution assumes a clearer meaning than in (5.5). It represents the total flow within the surface layer and ambiguities related to the analytical extension of the velocity field are avoided.

5.2. *The dynamic boundary condition*

The analysis of the dynamic boundary condition for a discontinuous, turbulent interface requires estimation of the mean stresses at a non-fluctuating level, which is done by integrating the components of the momentum equation in a direction

normal to the mean surface. In order to obtain the dynamic boundary condition at the base of the layer we first separate equation (4.15) for the conservation of linear momentum into equations for the two velocity components, $(\mathscr{U}, \mathscr{V})$, in the $\hat{\boldsymbol{s}}$- and $\hat{\boldsymbol{n}}$-directions respectively. Using the simplification of inviscid quasi-two-dimensional flow then gives

$$\frac{\partial\langle\mathscr{U}I\rangle}{\partial t}+\frac{\partial\langle\mathscr{U}^2I\rangle}{\partial s}+\frac{\partial\langle\mathscr{U}\mathscr{V}I\rangle}{\partial n}=\gamma\boldsymbol{g}\cdot\hat{\boldsymbol{s}}-\frac{1}{\rho}\left(\frac{\partial\langle\mathscr{P}I\rangle}{\partial s}+\frac{\Phi_1}{\rho}\right), \tag{5.13}$$

$$\frac{\partial\langle\mathscr{V}I\rangle}{\partial t}+\frac{\partial\langle\mathscr{V}^2I\rangle}{\partial n}+\frac{\partial\langle\mathscr{U}\mathscr{V}I\rangle}{\partial s}=\gamma\boldsymbol{g}\cdot\hat{\boldsymbol{n}}-\frac{1}{\rho}\left(\frac{\partial\langle\mathscr{P}I\rangle}{\partial n}+\frac{\Phi_2}{\rho}\right). \tag{5.14}$$

We integrate in the normal direction ($\hat{\boldsymbol{n}}$) between the two extremes $n=b$ and $n=h$. Straightforward integration of equations (5.13) and (5.14) and use of definitions (5.9) and (5.10) gives

$$\int_b^h\frac{\partial\langle\mathscr{U}I\rangle}{\partial t}\mathrm{d}n+\int_b^h\frac{\partial\langle\mathscr{U}^2I\rangle}{\partial s}\,\mathrm{d}n+\langle\mathscr{U}\mathscr{V}I\rangle|_h-\langle\mathscr{U}\mathscr{V}I\rangle|_b$$
$$=\boldsymbol{g}\cdot\hat{\boldsymbol{s}}d-\frac{1}{\rho}\int_b^h\left(\frac{\partial\langle\mathscr{P}I\rangle}{\partial s}+\Phi_1\right)\mathrm{d}n \tag{5.15}$$

and

$$\int_b^h\frac{\partial\langle\mathscr{V}I\rangle}{\partial t}\mathrm{d}n+\langle\mathscr{V}^2I\rangle|_h-\langle\mathscr{V}^2I\rangle|_b+\int_b^h\frac{\partial\langle\mathscr{U}\mathscr{V}I\rangle}{\partial s}\mathrm{d}n$$
$$=\boldsymbol{g}\cdot\hat{\boldsymbol{n}}d-\frac{1}{\rho}\left[\langle\mathscr{P}I\rangle|_h-\langle\mathscr{P}I\rangle|_b+\int_b^h\Phi_2\mathrm{d}n\right] \tag{5.16}$$

for the tangential and normal components respectively. Here Φ_1 and Φ_2 are the interfacial contributions:

$$\Phi_1=-\left\langle\mathscr{P}\frac{\partial I}{\partial s}\right\rangle,\quad \Phi_2=-\left\langle\mathscr{P}\frac{\partial I}{\partial n}\right\rangle. \tag{5.17}$$

These cannot be explicitly written in terms of the zone mean values and fluctuations unless some sort of closure assumption is made.

With the Leibnitz formula for the differential of integrals, equations (2.8) and (2.15) for the conditioned average within the liquid phase, the assumption of quasi-two-dimensional flow, $\gamma|_h=0, \gamma|_b=1$ and the kinematic boundary condition (5.11) we obtain an averaged boundary condition for the tangential component of stress, or momentum transfer. After some rearrangement the final result is

$$\rho\left[\langle uv\rangle|_b+U|_bW-\frac{\partial b}{\partial s}\left(\frac{P|_b}{\rho}+\langle u^2\rangle|_b\right)\right]$$
$$=\rho\left\{\frac{\partial}{\partial t}\int_b^h\gamma U_w\mathrm{d}n+\frac{\partial}{\partial s}\int_b^h\gamma\left[U_w^2+\frac{P_w}{\rho}+\langle u^2\rangle_w\right]\mathrm{d}n-\boldsymbol{g}\cdot\hat{\mathbf{s}}\,\mathrm{d}\right\}+\int_b^h\Phi_1\mathrm{d}n. \tag{5.18}$$
$$\text{(I)}\qquad\qquad\text{(II)}\qquad\qquad\text{(III)}\qquad\text{(IV)}$$

Each side of the boundary condition represents the flux of tangential momentum through the surface $n=b$. Alternatively, the right-hand side can be seen to give an equation for the motion of a layer above the bottom boundary $n=b$. By comparing

it with the equation for a thin layer of fluid over a variable bed the contributions can be interpreted as:

(I) tangential acceleration of the mean flow;
(II) flux of momentum within the layer (i.e. including turbulent velocity fluctuations);
(III) the component of the weight of water within the layer acting to increase U_w;
(IV) the interfacial terms: contributors to this term are the air drag acting on drops of water, and the inertial effects of bubbles, but their effect where air and water interpenetrate is largely unknown.

We operate in similar fashion on the normal equation to obtain

$$P|_b + \rho\left[\langle v^2\rangle|_b + V|_b W - \langle uv\rangle|_b \frac{\partial b}{\partial s}\right]$$

$$= \rho\left\{-\boldsymbol{g}\cdot\hat{\boldsymbol{n}}d + \frac{\partial}{\partial t}\int_b^h \gamma V_w \mathrm{d}n + \frac{\partial}{\partial s}\int_b^h \gamma(U_w V_w + \langle uv\rangle_w)\mathrm{d}n\right\} + \int_b^h \Phi_2 \mathrm{d}n. \tag{5.19}$$

(I) (II) (II) (III)

The left-hand side includes both the mean pressure at the base of the layer and the contributions due to fluxes of normal momentum through the base of the layer. Note that contributions arising from the interaction between the mean field and the turbulent fluctuations can be regarded as 'integrated interaction stresses' (Hasselmann 1971). The contributions appearing on the right-hand side can be interpreted as:

(I) the weight of the fluid within the layer;
(II) the change of normal momentum of the layer due to both the mean and turbulent flow;
(III) the interfacial term.

Thus we see that each of the boundary conditions can also be interpreted as either a boundary condition or an integral equation for the surface layer, and the other side of the equation then gives the mass and momentum exchanges with the surface layer or the bulk liquid respectively. As stated above, viscous terms have been omitted.

Note that, under the assumption of negligible mean velocity in the normal direction within the layer, and negligible interfacial terms, the normal component of the momentum equation can be re-written as

$$\frac{P|_b}{\rho} + \langle v^2\rangle|_b + V|_b W - \langle uv\rangle|_b \frac{\partial b}{\partial s} \approx -\boldsymbol{g}\cdot\hat{\boldsymbol{n}}d + \frac{\partial}{\partial s}\int_b^h \gamma\langle uv\rangle_w \mathrm{d}n, \tag{5.20}$$

which simply relates the pressure and the fluxes of normal momentum at the base of the layer to the weight of water within the layer and the gradient of tangential turbulent stresses.

5.3. *The boundary condition for k*

For a continuous, turbulent interface the boundary condition for k at the mean free surface is most often given in terms of the normal derivative:

$$\left[\frac{\partial k}{\partial n}\right]_{\bar{\eta}} = 0. \tag{5.21}$$

Here we analyse the full equation representing the appropriate boundary condition in the case of a discontinuous, turbulent interface. We first expand equation (4.23) for

the components and continue the restriction to quasi-two-dimensional flow to reduce the number of terms shown explicitly:

$$\frac{\partial(\gamma k)}{\partial t}+\frac{\partial(\gamma k U_w)}{\partial s}+\frac{\partial(\gamma k V_w)}{\partial n}+\gamma\left[\langle u^2\rangle_w\frac{\partial U_w}{\partial s}+\langle uv\rangle_w\left(\frac{\partial U_w}{\partial n}+\frac{\partial V_w}{\partial s}\right)+\langle v^2\rangle_w\frac{\partial V_w}{\partial n}\right]$$
$$+\left[\frac{\partial(\gamma\langle uk\rangle_w)}{\partial s}+\frac{\partial(\gamma\langle vk\rangle_w)}{\partial n}\right]+\frac{1}{\rho}\left[\frac{\partial(\gamma\langle pu\rangle_w)}{\partial s}+\frac{\partial(\gamma\langle pv\rangle_w)}{\partial n}\right]$$
$$+\frac{\gamma P_w}{\rho}\left[\frac{\partial U_w}{\partial s}+\frac{\partial V_w}{\partial n}\right]-\nu\frac{\partial}{\partial s}\left[\frac{\partial(\gamma k)}{\partial s}+\frac{\partial(\gamma\langle u^2\rangle_w)}{\partial s}+\frac{\partial(\gamma\langle uv\rangle_w)}{\partial n}\right]$$
$$-\nu\frac{\partial}{\partial n}\left[\frac{\partial(\gamma k)}{\partial n}+\frac{\partial(\gamma\langle v^2\rangle_w)}{\partial n}+\frac{\partial(\gamma\langle uv\rangle_w)}{\partial s}\right]$$
$$-\nu\gamma\left[\left(\frac{\partial U_w}{\partial s}+\frac{\partial V_w}{\partial n}\right)^2+U_w\frac{\partial^2 U_w}{\partial s^2}+U_w\frac{\partial^2 V_w}{\partial s\partial n}+V_w\frac{\partial^2 U_w}{\partial s\partial n}+V_w\frac{\partial^2 V_w}{\partial n^2}\right]$$
$$+\frac{\Psi}{\rho}+\Upsilon+\gamma\epsilon=0. \tag{5.22}$$

To obtain the required boundary condition we use the same method as given in the previous sections, i.e. we integrate the above equation between the two non-fluctuating levels $n=b$ and $n=h$. Here two reasonable extra conditions are imposed such that $k|_h=0$ and $\langle u_iu_j\rangle_w|_h=0$. Using the kinematic boundary condition (5.11) and Leibnitz' rule we obtain

$$\left[\langle vk\rangle+\frac{\langle pv\rangle}{\rho}+kW-\nu\left(\frac{\partial(\gamma\langle uv\rangle)}{\partial s}+\frac{\partial\gamma(k+\langle v^2\rangle)}{\partial n}\right)\right]_b-\left[\langle uk\rangle+\frac{\langle pu\rangle}{\rho}\right]_b\frac{\partial b}{\partial s}$$
$$+\left[\nu\left(\frac{\partial\gamma(k+\langle u^2\rangle)}{\partial s}+\frac{\partial(\gamma\langle uv\rangle)}{\partial n}\right)\right]_b\frac{\partial b}{\partial s}=\frac{\partial}{\partial t}\int_b^h\gamma k\,\mathrm{d}n+\frac{\partial}{\partial s}\int_b^h\gamma kU_w\,\mathrm{d}n$$
$$+\int_b^h\gamma\left[\langle u^2\rangle_w\frac{\partial U_w}{\partial s}+\langle uv\rangle_w\left(\frac{\partial U_w}{\partial n}+\frac{\partial V_w}{\partial s}\right)+\langle v^2\rangle_w\frac{\partial V_w}{\partial n}\right]\mathrm{d}n+\frac{\partial}{\partial s}\int_s^h\gamma\left[\langle uk\rangle_w+\frac{\langle pu\rangle_w}{\rho}\right]\mathrm{d}n$$
$$+\frac{1}{\rho}\int_b^h\gamma P_w\left(\frac{\partial U_w}{\partial s}+\frac{\partial V_w}{\partial n}\right)\mathrm{d}n-\nu\frac{\partial}{\partial s}\int_b^h\left[\frac{\partial(\gamma k)}{\partial s}+\frac{\partial(\gamma\langle u^2\rangle_w)}{\partial s}+\frac{\partial(\gamma\langle uv\rangle_w)}{\partial n}\right]\mathrm{d}n$$
$$-\nu\int_b^h\gamma\left[\left(\frac{\partial U_w}{\partial s}+\frac{\partial V_w}{\partial n}\right)^2+U_w\frac{\partial^2 U_w}{\partial s^2}+U_w\frac{\partial^2 V_w}{\partial s\partial n}+V_w\frac{\partial^2 U_w}{\partial s\partial n}+V_w\frac{\partial^2 V_w}{\partial n^2}\right]\mathrm{d}n$$
$$+\frac{1}{\rho}\int_b^h\Psi\,\mathrm{d}n+\int_b^h(\Upsilon+\gamma\epsilon)\mathrm{d}n. \tag{5.23}$$

The left-hand sides of these equations contain all the boundary contributions from the base of the layer. These are derived by integrating the diffusion due to velocity

fluctuations and the working of pressure fluctuations. They can be also regarded as the flux of turbulent kinetic energy due to the entrainment W, kW, to pressure fluctuations, $\langle vp \rangle / \rho$, and to turbulent convection, $\langle vk \rangle$. Diffusive viscous terms depend on the gradients of the turbulence. The flux of turbulent kinetic energy across $n = b$ equals a number of contributions which are integrated over the layer thickness. The nature of the various integrands can be interpreted from the discussion in § 4.3.

The above equations form a complicated set and give the general boundary condition to be applied at the base of the layer $n = b$. For practical applications and for models which aim at analysis of the global system waves, turbulent flow in the water and in the surface layer – it is necessary to use appropriate closures and assumptions on the flow and to consider only the most relevant contributions.

6. Closure

The major difficulty in using averaged equations to describe turbulence is the closure problem: finding adequate approximations for the Reynolds stresses etc. Now the above averaged boundary conditions (5.11), (5.18), (5.19) and (5.23) introduce on their, right-hand sides a further range of terms that need to be determined. First, consider the structure of the problem.

The number of boundary conditions for a liquid at a free surface is one greater than that required at a rigid surface since the position of the surface is an extra independent variable that must be determined as part of the problem. Here there are numerous extra variables, of which the major/simpler ones are:

b :	the position of the base of the layer;
d :	the 'equivalent thickness' of the layer;
W :	the mass transfer to the layer;
$\int_b^h \gamma U_w \, \mathrm{d}n$:	the mass flow in the layer;
also	the terms representing momentum flow in the layer;
	the terms representing phase interaction in the layer,
	e.g. air drag, capillarity effects, etc.

Note that these terms include the mean flow properties of the surface layer, as well as the turbulent fluctuations within the layer.

As usual, we cannot use fundamental conservation concepts for our modelling unless we return to the full equations of motion. Our discussion in Part 1 of the various regimes and flows when turbulence meets a free surface provides the key to developing closures, since for many cases the properties of the turbulence below the surface determines the character of the surface layer. For example, as in Part 1, we may represent the turbulence by its kinetic energy density and a representative length scale. Then we can identify the type of flow regime and propose closures accordingly.

In the general study of turbulence in a single fluid there is a wealth of experimental evidence to guide the creation of closure equations and to check their value. We have found almost no experimental work where turbulent parameters are directly available to quantify the different regimes, and very few works that give information on properties of the surface layer. Thus, for the present we give only a limited simple introduction to closure in the next section.

A further complication is that the above approach is inappropriate where the turbulence is directly generated within the surface layer. The surface-layer boundary condition for k is equation (5.23). There are terms that may be responsible for production on the right-hand side. Bearing in mind that typical turbulent fluctuations

are large in breaking waves it seems that the major production terms may be

$$\int_b^h \gamma \left[\langle u^2 \rangle_w \frac{\partial U_w}{\partial s} + \langle uv \rangle_w \frac{\partial U_w}{\partial n} \right] \mathrm{d}n$$

since V_w and its gradients may be much smaller than those of the tangential mean velocity component.

Additionally, perhaps the most potent source of turbulence occurs at the foot of a breaker when the water falling, tumbling, or sliding, down the front meets the undisturbed water with a significantly different velocity. This aspect of the problem is discussed further in § 8.

7. Discussion of simple closures

The model used for the underlying single-phase liquid requires closures for the turbulent terms in the averaged equations of motion. For present purposes we presume that some closure has been chosen for the underlying liquid with sufficient complexity to give those properties of the base of the surface layer, indicated with subscript b, that are needed here. As indicated in Part 1, we suppose that primary properties of the free surface depend on turbulent kinetic energy k, or q, and a turbulent length scale L. This (k, L) assumption also serves to polarize this discussion into consideration of two extreme cases, although many intermediate examples occur.

7.1. *A passive surface layer*

The first of these cases, and the simplest, is where the structure of the free surface depends entirely on turbulence reaching it from below. That is, the surface layer is roughly horizontal and we assume that its properties are just due to the turbulence at its base. In this case all the layer terms needed for closure might be estimated from a knowledge of the flow's location in the (q, L)-plane, as in figure 10 of Part 1, and the major effect at the surface is due to gradients of the turbulent properties along the surface.

For flows driven by turbulence from below, we assume that some of the mean and turbulent flow properties have a uniform profile across the surface layer. This is reasonable if interaction with the air is neglected. We take

$$U_w(s, n) \approx U|_b(s), \quad V_w(s, n) \approx V|_b(s). \tag{7.1}$$

However, other variables describing the turbulence, such as $\langle v^2 \rangle_w$, must have a gradient across the layer since they are zero at the outer boundary of the surface layer. To simplify our analysis, and due to a lack of reliable information, we use the crossflow structure, $f(n)$, which is the same for each of $\langle v^2 \rangle_w$ and $\langle uv \rangle_w$ and ϵ, giving

$$\left.\begin{aligned} \langle u^2 \rangle_w(s, n) \approx \langle u^2 \rangle|_b(s),\ \langle v^2 \rangle_w(s, n) \approx \langle v^2 \rangle|_b(s) f(n),\ \langle uv \rangle_w(s, n) \approx \langle uv \rangle|_b(s) f(n), \\ k(s, n) \approx \langle u^2 \rangle|_b(s) + \langle v^2 \rangle|_b(s) f(n), \quad \epsilon(s, n) \approx \epsilon|_b(s) f(n). \end{aligned}\right\} \tag{7.2}$$

The mean pressure might be hydrostatic:

$$P_w \approx \rho \boldsymbol{g} \cdot \hat{\boldsymbol{n}} \int_n^h \gamma \, \mathrm{d}n$$

with the fluctuation in pressure, p_w, perhaps being small enough to ignore. Gravity also influences v_w. For the present we just suppose that terms involving v_w and,

consequently, turbulence terms of equation (7.2) vary linearly to zero across the layer, i.e.

$$f(n) = \frac{h-n}{h-b}, \quad \text{which gives} \quad f(b) = 1 \quad \text{and} \quad f(h) = 0. \tag{7.3}$$

With these assumptions the kinematic and dynamic boundary conditions become

$$\frac{\partial b}{\partial t} + U|_b \frac{\partial b}{\partial s} - V|_b = -W = -\left(\frac{\partial d}{\partial t} + d\frac{\partial U|_b}{\partial s}\right), \tag{7.4}$$

$$\langle uv\rangle|_b + U|_b W - \frac{\partial b}{\partial s}\left(\frac{P|_b}{\rho} + \langle u^2\rangle|_b\right)$$

$$= \frac{\partial}{\partial t}(dU|_b) + dU|_b\frac{\partial U|_b}{\partial s} + \boldsymbol{g}\cdot\hat{\boldsymbol{n}}\frac{\partial \Gamma_p}{\partial s} + \frac{\partial}{\partial s}(d\langle u^2\rangle|_b) - \boldsymbol{g}\cdot\hat{\boldsymbol{s}}d, \tag{7.5}$$

$$\frac{P|_b}{\rho} + \langle v^2\rangle|_b + V|_b W - \langle uv\rangle|_b\frac{\partial b}{\partial s}$$

$$= -\boldsymbol{g}\cdot\hat{\boldsymbol{n}}d + \frac{\partial}{\partial t}(dV_b) + \frac{\partial}{\partial s}(dU_bV_b) + \frac{\partial}{\partial s}[\Gamma_1(h-b)\langle uv\rangle_b]. \tag{7.6}$$

We refrain from giving the boundary condition for k_b. It may be obtained from the general boundary condition for k, equation (5.23), if similar simple assumptions are made about the triple correlations.

These boundary conditions are still not fully closed. As well as the quantities at the base of the layer, which must relate to the modelling of the bulk liquid, they contain a number of layer properties that may be taken to depend on the particular flow regime. These are d, $(h-b)$, Γ_p, and Γ_1. The Γ are such that

$$\Gamma_p = \int_b^h \gamma(n)\left[\int_n^h \gamma(\zeta)d\zeta\right]\mathrm{d}n, \quad \Gamma_1 = \frac{1}{h-b}\int_b^h \gamma(n)f(n)\,\mathrm{d}n. \tag{7.7}$$

More such integrals appear in the boundary condition for k.

The natural approach, now, is to relate the amount of water in the layer (or layer equivalent thickness), d, and the actual thickness of the layer, $(h-b)$, to the turbulence q through the typical turbulent length scale L of the blobs of water as follows:

$$d = \alpha L(q), \quad (h-b) = \beta L(q). \tag{7.8}$$

Here α and β are non-negative proportionality functions which depend on the flow regime under consideration, that is they depend on the values (q, L) that characterize the turbulence as suggested in Part 1. Note that the ratio $d/(h-b)$ is the average value of γ over the surface layer: it should be noted that although this ratio is a useful concept for making numerical estimates of these quantities and the Γ, it is less useful for any calculations since both d and $(h-b)$ are zero for a flat surface. Hence α and β are both zero for small Froude and Weber numbers, or $q \to 0$.

Finally, suitable values of α and β can be roughly estimated for each flow regime. We now use some of the surface models in § 2, keeping in mind that $\alpha = 0$ and $\beta = 0$ for a flat surface and they are both expected to increase with the intensity of turbulence.

For wavy surfaces β corresponds to twice the wave steepness defined as aK where a is wave amplitude and K is wavenumber, $K = 1/L$ being a reasonable estimate for the typical surface disturbance. For nearly two-dimensional waves, this gives $d = a$,

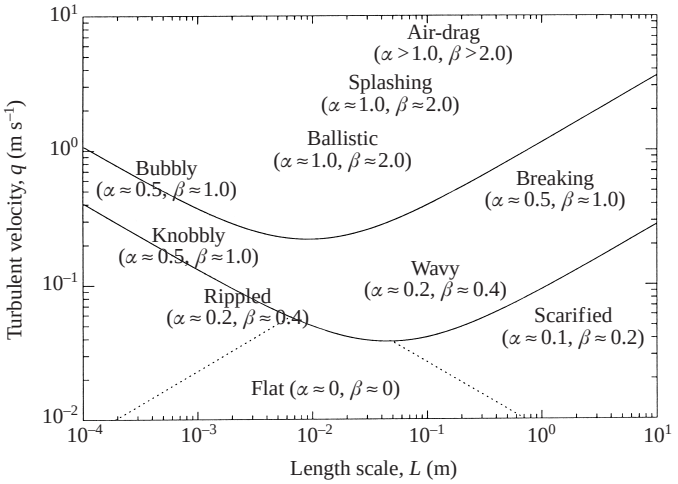

FIGURE 5. Diagram of the (L, q)-plane with tentative estimate of the constants α and β of equation (7.8) for some flow regimes.

and $\alpha = \frac{1}{2}\beta = aK$ and a somewhat lower value for very steep waves. On the other hand, for three-dimensional waves different values occur, e.g. two equal sinusoidal wave trains at right angles to each other, each with wavenumber K and amplitude a, could arise from the same value of L, but have $\beta = 4aK$ and $\alpha = 2aK$. This suggests that for the gravity-dominated regime values $\alpha \approx 0.2$ and $\beta \approx 0.4$ may be the upper limits for wavy flows.

For scarified flow using the surface model of equation (3.6), and the intermittency factor from equation (3.7), we obtain

$$d(s,t) \equiv \int_b^h \gamma \, \mathrm{d}n = \frac{m}{m+1}(h-b) \tag{7.9}$$

in which m is the order of the polynomial function used to represent the scar. In Part 1 (§ 3) we assumed a turbulence length of 5 times the scar thickness $(h-b)$. Then equation (7.9) gives a relation between the amount of water d and the turbulence length $L(q)$:

$$\alpha = \frac{d}{L(q)} = \frac{m(h-b)/(m+1)}{5(h-b)} = \frac{m}{5(m+1)}, \quad \beta = \frac{h-b}{L(q)} = \frac{(h-b)}{5(h-b)}, \tag{7.10}$$

thus for the case $m = 4$, $\alpha = 0.16$ and $\beta = 0.20$. For this regime large values of L lead to more widely spaced less significant scars and $\alpha \to \beta \to 0$. The same also happens for smaller values of q, leaving essentially flat surfaces. Similar arguments can be used to assess rippled and knobbly flows with values as shown in figure 5.

For splashing flows we infer that α is likely to become larger, perhaps even larger than unity for violently splashing surfaces, with β being about double α, or even slightly more. Note our argument in Part 1 that fluctuating turbulent velocities perpendicular to the surface could be bigger than the tangential velocities. The amount of water contained in a splashing surface layer can be estimated by integrating the intermittency factor (3.3):

$$d(s,t) \equiv \int_{\bar{\eta}-3\sigma}^{\bar{\eta}+3\sigma} \frac{1}{2}\left[-\mathrm{erf}\left(\frac{n-\bar{\eta}(s,t)}{\sqrt{2}\sigma(s,t)}\right)\right] \mathrm{d}n = 3\sigma \tag{7.11}$$

in which we assume all the surface layer to be within a distance of 3σ from the mean free surface $\bar{\eta}$ (layer width of 6σ). The symmetry indicates a near cancellation of the effects of drops and bubbles in such flows. For examples in between a connected surface and a fully splashing case we can adopt a monotonically increasing value of α such that for breaking values of $\alpha \approx 0.5$ and $\beta \approx 1.0$ might be used.

All the above information can be summarized in a length–velocity diagram similar to those of Part 1 as shown in figure 5.

The above discussion indicates how a closure might be constructed, for example we find $0.3 \leqslant \Gamma_1 \leqslant 0.5$. It also indicates two other aspects of this, the simplest problem, where well-defined turbulence meets a free surface. One is the paucity of experimental data on the surface and its fluctuations. Secondly, the effect of a gradient of turbulence properties leads to variations in the terms in the boundary conditions which in turn may lead to a turbulence-generated mean flow, e.g. Hong & Walker (2000).

7.2. *A simple model for a splashing free surface*

The simplest case we can imagine is where the splashing and spray is simply in free fall and we ignore all entrainment of bubbles. We model the disturbed free surface as being made up of a set of drops which are all ejected from the base of the surface layer, e.g. N drops per unit area with a given distribution of velocities, so we can then work out the various terms in the boundary conditions, by supposing the drops fall under gravity and do not interact.

A slight problem arises, so we illustrate it by first considering drops which are ejected upwards uniformly from a horizontal surface with velocity V_0. For a drop ejected at time $t = -t_0$, we know its velocity

$$V = V_0 - g(t + t_0) \tag{7.12}$$

and height

$$n = V_0(t + t_0) - \tfrac{1}{2}g(t + t_0)^2. \tag{7.13}$$

The maximum height of drops is $n = H = V_0^2/2g$. Thus if we wish to know the distribution of drops at $t = 0$, in order to find γ for example, we have

$$n = b + V_0 t_0 - \tfrac{1}{2}g t_0^2 = b + t_0(V_0 - \tfrac{1}{2}g t_0) \tag{7.14}$$

for all t_0 in $0 < t_0 < 2V_0/g$.

Drops in the interval $(n, n + \Delta n)$ start in the time interval $(-t_0 - \Delta t_0, -t_0)$ so

$$\Delta n = (V_0 - g t_0)\Delta t_0. \tag{7.15}$$

Let the total number of particles ejected per unit time per unit area, be N and let the mass of each be m. (The mass too could have a distribution, change with time if evaporating or condensing, and have a drag dependent on mass/or volume, but we avoid extra complication.) Then, the mass ejected in time Δt_0 is $mN\Delta t_0$ giving a density of $mN\Delta t_0/\Delta n$ at $n = mN/(V_0 - g t_0) = \rho\gamma$ but $V_0 - g t_0 = \pm\sqrt{V_0^2 - 2gn}$ so, noting both signs, $+$ and $-$, for drops going up and down respectively, we gain a factor 2, giving

$$\rho\gamma = \frac{2mN}{\sqrt{V_0^2 - 2gn}}. \tag{7.16}$$

This gives us a problem due to the singularity at $n = H$.

The difficulty is a result of applying continuum ideas, i.e. $\Delta t_0, \Delta n \to 0$, to a discrete phenomenon. However, we can avoid the difficulty by noting that all we need are

integrals over the layer and that this singularity is integrable. Hence

$$\int_0^H \rho\gamma \, \mathrm{d}n = 2mNV_0/g, \tag{7.17}$$

as may be deduced from the time of flight of each drop.

Now, consider particles ejected from a plane surface with velocities (U_0, V_0), with probability $f(U_0, V_0|s_0, t_0)$, i.e. s_0 measures their position along the surface and f can vary with position and time. It is a probability density function so

$$\int_0^{\max V_0} \int_{\min U_0}^{\max U_0} f(U_0, V_0|s_0, t_0) \, \mathrm{d}U_0 \, \mathrm{d}V_0 = 1. \tag{7.18}$$

We let the surface have a downward slope θ so that

$$\boldsymbol{g} = g(\sin\theta, -\cos\theta). \tag{7.19}$$

Now choose a position (s, n), above the surface, and if t_1 is now the time of flight of a particle at the time of interest

$$s = s_0 + U_0 t_1 + \tfrac{1}{2} g \sin\theta t_1^2, \quad n = b + V_0 t_1 - \tfrac{1}{2} g \cos\theta t_1^2 \tag{7.20a,b}$$

$$U = U_0 + g \sin\theta t_1, \quad V = V_0 - g\cos\theta t_1. \tag{7.20c,d}$$

Thus if we choose (s, n) and (U_0, V_0) these equations give two values for each of t_1, U, V, s_0. Then, any property is found by the appropriate integration of $f(U_0, V_0|\, s_0, t - t_1) \, \mathrm{d}U_0 \, \mathrm{d}V_0$ and the further integration over n to avoid the square-root singularity.

We find that $g\cos\theta t_1 = V_0 \mp V_1$ where $V_1^2 = V_0^2 - 2g(n-b)\cos\theta$. Therefore

$$V = \pm V_1 \text{ and } U = U_0 + V_0 \tan\theta \pm V_1 \tan\theta, \tag{7.21}$$

where $+$ is for drops going up, and $-$ is for drops falling down. Since we suppose we know U_0, V_0 the argument, following the initial example, now goes as follows.

For time Δt_1 mass $mN\Delta t_1 \Delta s_0$ is ejected from Δs_0 and the corresponding volume $\Delta n \Delta s$ at s, n is obtained from the relations for (s, n) as

$$\Delta s = (U_0 - g\sin\theta t_1)\Delta t_1 = U\Delta t_1 \tag{7.22}$$

and

$$\Delta n = (V_0 - g\cos\theta t_1)\Delta t_1 = V\Delta t_1. \tag{7.23}$$

Hence the contribution to the local 'density' $\rho\gamma_1$ is $\rho mN\Delta t_1 \Delta s_0/(UV\Delta t_1^2)$ and $\Delta s_0 = -U_0\Delta t_1$ which implies $\gamma_1 = \rho mNU_0/(UV)$, where the magnitude of the velocities should be used. Thus

$$\gamma_1 = \frac{mN}{V_1} = \frac{mN}{\sqrt{V_0^2 - g(n-b)\cos\theta}} \tag{7.24}$$

for drops going up and drops coming down. But this γ_1 is only for one (U_0, V_0), hence

$$\gamma = \int_{\sqrt{g(n-b)\cos\theta}}^{\max V_0} \int_{\min U_0}^{\max U_0} 2\gamma_1 f(U_0, V_0|s_0, t - t_1) \, \mathrm{d}U_0 \, \mathrm{d}V_0. \tag{7.25}$$

Similar integrals can be deduced for all the other properties of the layer, which can then be fed into the various terms needed for 'closing' the boundary conditions. Then one can investigate particular forms of the probability distribution f. We have little information to guide us on f. Projection velocities near zero might be discounted

as irrelevant, and independent distributions for U_0 and V_0, which would simplify integration, are seen to be inappropriate as they give a bias in the velocity space: for example, if each component is taken to have a uniform probability over a finite interval of velocities the resulting distribution is non-zero in a rectangular region of velocity space. Radial variation of probability in velocity space seems more appropriate. There is some information available on drop distributions, e.g. see Andreas *et al.* (1993), but most of the data relate to drops that are not very close to the surface.

The above discussion shows some of the considerations that are needed for a better understanding of the surface layer. The splashing case is very simple in concept with no interaction between drops, whereas all real flows are more complex and include bubbles and mixed regions. On the other hand some details in the layer may be of little importance as far as the boundary conditions are concerned.

In this pure splashing case simplifications can be made. For example the tangential momentum transfer to the bottom of the layer is clearly one of the more important parameters. To evaluate it we just need to know the increased tangential velocity component of drops as they rejoin the bulk liquid. From equations (7.20), the end of the trajectory is at $t_1 = 2V_0/g\cos\theta$ with a change in U of $2V_0\tan\theta$. Thus the average rate of tangential momentum flux across the base of the layer is

$$2mN\tan\theta\int_0^{\max V_0}\int_{\min U_0}^{\max U_0} V_0 f(U_0, V_0|s_0, t-t_1)\,\mathrm{d}U_0\,\mathrm{d}V_0. \tag{7.26}$$

Note that the integral as written conceals the dependence of t_1 and s_0 on U_0 and V_0. However, if the conditions along the surface are independent of space and time, the integral is just the first moment of V_0.

Similarly the normal flux of normal momentum may be considered from first principles. The velocity fluctuations at $n = b$ include those which cause the drop ejection, i.e. create momentum mV_0 in the normal direction for each drop. Similarly on impact each returning drop gives up the same momentum of the opposite sense. In terms of the effective normal stress they both contribute in the same sense so that the normal stress at the base of the surface layer is

$$mN\int_0^{\max V_0}\int_{\min U_0}^{\max U_0} V_0 f(U_0, V_0|s, t)\,\mathrm{d}U_0\,\mathrm{d}V_0$$
$$+mN\int_0^{\max V_0}\int_{\min U_0}^{\max U_0} V_0 f(U_0, V_0|s_0, t-2V_0/g\cos\theta)\,\mathrm{d}U_0\,\mathrm{d}V_0. \tag{7.27}$$

This expression is directly related to the corresponding tangential expression. For the uniform case, i.e. f independent of s and t, both terms in (7.27) are identical and are also equal to the normal component of the weight per unit area of the drops along the normal to the surface.

Note how the dynamic effect of splashing when there is no interaction between drops, or with the air, is essentially related to the acceleration of the drops under gravity, i.e. the weight of the water that is in the air. In addition, the influence that can arise from higher up a breaker slope is shown by the dependence on s_0 and t_1 in (7.26) and (7.27). Further, the effect of air drag would be to reduce the velocities at which drops return to the bulk liquid and thus the general effect of air on the water dynamics in the splashing regime is to support some of weight of splashing water, though this may actually balance out with a greater air pressure. The above

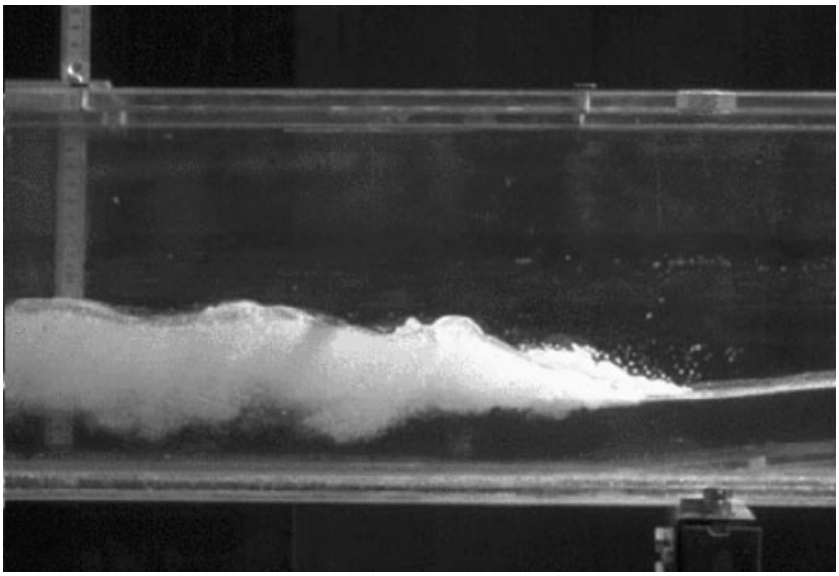

FIGURE 6. Photograph of a hydraulic jump. This is the same flow as shown in Part 1 figure 5, but in this case there was a layer of tiny bubbles on top of the water entering the breaker. The bubbles were sufficiently small that they took more than five minutes to rise 10 cm in water at rest, thus they illustrate the extent of the turbulence-induced mixing of the surface water. The vertical scale on the left is marked in cm. The incoming flow is about 4 cm deep.

discussion is probably too strong an idealization since in practice splashing leads to bubble formation, and flow in the bubbly region should also be modelled.

8. Breaker toe dynamics

The masses of water falling, tumbling and sliding down the front of a spilling breaker or bore fall onto the smooth incoming water at the leading edge of the breaker, which is where the major generation of turbulence takes place (see figure 6). Traditionally this is usually envisaged as a 'roller' riding on the front of a wave meeting the smooth water at its foot. Peregrine & Svendsen (1978) pointed out, in a paper that stimulated or informed a number of experimental measurements (Battjes & Sakai 1981; Stive 1984) that the turbulent flow in a spiller cannot easily be separated into a 'roller' and the rest. There is a stream of turbulence initiated at the foot of the spiller, from which a roller can only be divided once mean streamlines are determined. Also the fluctuating velocities are just as large as the mean velocities. Hence, here, we do not use the term 'roller', though the concept is useful in indicating forward flow in the surface layer down the face of a breaker. We use 'breaker' instead. We are keen to be able to model unsteady waves, and for such waves there is then no unambiguous way of determining a separating mean streamline to define a roller's boundary.

There is another piece of traditional terminology that it seems advisable to change. The forward edge of a breaker's turbulent splashing front is sometimes referred to as the 'toe' of the breaker. Measurements, especially of hydraulic jumps, show a continuous mean surface through the breaker at this point. However, there are problems in defining a mean surface when unsteady irregular flows move over a smooth surface. This has been discussed in Brocchini & Peregrine (1996, § 3) for the swash zone on a beach and the same ideas hold in this problem. If one considers the

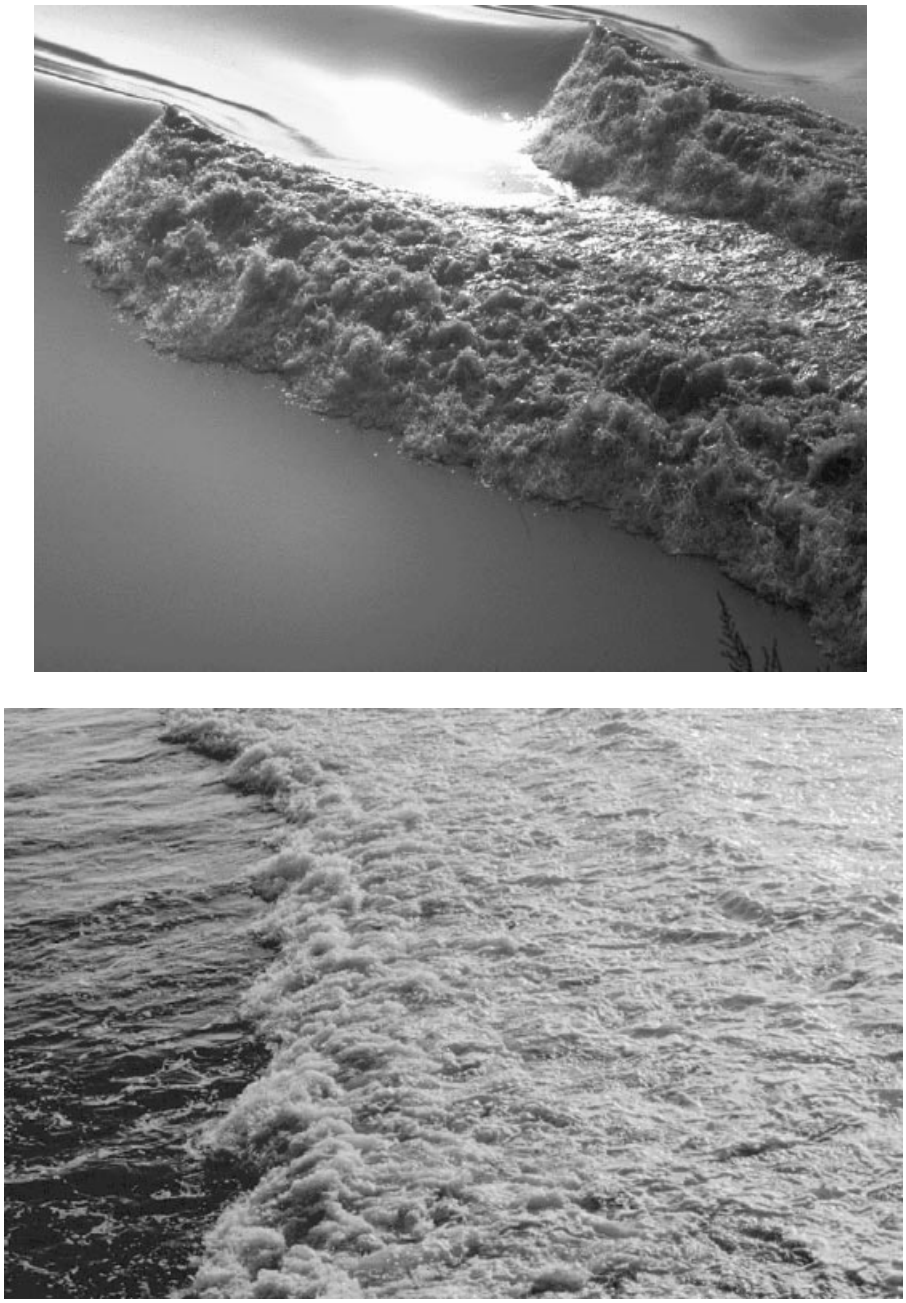

FIGURE 7. Typical examples of 'toes' occurring between smooth and turbulent flow at the foot of a river bore (top) and of a bore on a beach (bottom). In each case we can see about 3 m length of bore.

mean height of water as one moves through from smooth water to a breaker there is a smooth transition as mentioned. However, if one looks from above at the breaker's boundary it is not smooth and an averaging along the base of the breaker will give a mean position for the base of the breaker. We propose that a mean position for base of a breaker, however determined from the three-dimensional character of the

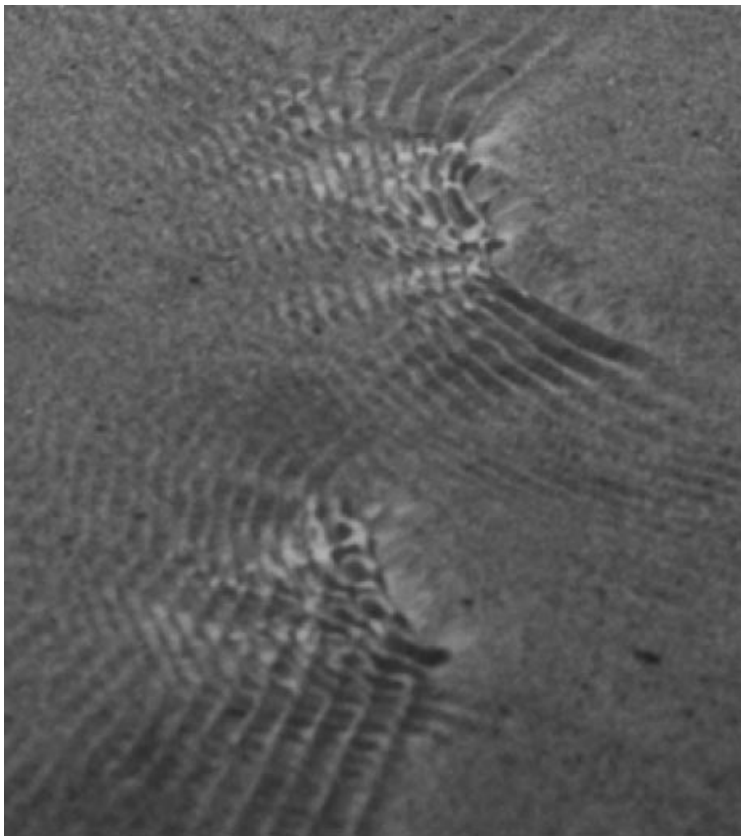

FIGURE 8. Spilling breakers at the smallest scale where they are nearly dominated by capillary waves. The photograph shows about 10 cm width of water.

turbulent flow, should be called the 'foot' of the breaker. In addition we propose that the fluctuations of the boundary between smooth and turbulent flow at the water surface be called the 'toes' of the breaker, see the examples in figure 7.

This apparently simple change of terminology brings a real benefit to the modelling of breakers. It is difficult to define the strength of a breaker, for example comparisons are often made with hydraulic jumps of the same height. We are unaware of any measure that gives an instantaneous strength for a breaker. The concept of a breaker foot, with toes fluctuating around it, immediately gives two parameters that can be used to measure the instantaneous 'strength' of a breaker: that is, the mean water thickness of the surface layer and the mass flow in the surface layer at the breaker's foot position. These are the quantities d and $\rho \int_b^h \gamma U_w \, \mathrm{d}n$ introduced above, though the mean velocity

$$U_m = \frac{1}{d} \int_b^h \gamma U_w \, \mathrm{d}n \tag{8.1}$$

may be a more useful parameter than the mass flow. This velocity gives a measure of the influence of the breaker's height. For example, in deep water intermittent breaking is very common. Typically this is due to the disparity in phase and group velocities leading to individual crests losing amplitude and consequently U_m decreases.

These parameters can be used with Peregrine & Svendsen's (1978) suggestion that the primary generation of turbulence occurs at the foot of a breaker in a similar

manner to a mixing layer. This is not quite the traditional mixing layer that has been the subject of many studies to clarify its turbulent structure, since the velocities of the two layers that come together at the breaker's foot are in opposing directions relative to the foot rather than in the same direction, or with one being at rest. Thus breaker turbulence is initiated in a more violent fashion, especially since one of the layers is already strongly turbulent. Indeed a measure of the size of the 'toes' may also be a useful strength parameter for this turbulence.

The value of defining a height and velocity at the foot of a breaker can be seen by considering the modelling of breaker flow in Svendsen *et al.* (2000) and Veeramony & Svendsen (2001). They consider the mean vorticity in the flow and take the thickness of the turbulent layer as zero at the foot of the breaker. The concepts developed here can give a more rational initial condition for their modelling of vorticity. In addition their modelling uses a zero-stress boundary condition at the mean free surface which may be improved by consideration of the results of § 5.

The 'foot and toes' concept seems sound at all scales. At large scales there is substantial splashing of blobs of water: this may be best seen by viewing a breaker from head-on close to the water level. At small scales surface tension restrains the blobs so that there is no splashing but as shown in Part 1 figure 5 there are still significant toes. It is only at the smallest scale where ripples might ride ahead of the breaker that the toes may be smoothed. Figure 6 of Part 1 shows such a case which seems to be on the margin of producing ripples, but is not doing so because of a surface active agent in the water. On the other hand figure 8 here shows very small breakers with ripples where the turbulence is just strong enough for an irregular foot to be perceived among the strong ripples.

9. Discussion

Following the descriptive and semi-quantitative discussion of Part 1, this paper gives an analytical approach to modelling a strongly disturbed free surface. Full recognition is given to the possibility of a two-phase region. Equations of motion are derived, *ab initio*, for that region. Although these equations of motion are not new, we need to define notation and we have used an integral approach for the derivation that seems better suited to such discontinuous flows than one based initially on differential equations. In the equation for the turbulent kinetic energy extra work terms appear, due both to the mean pressure and to the mean rate of strain, because of the integral procedure. These contributions, which are identically zero outside the surface layer, show that the air–water mixture within any control volume has some properties like a compressible fluid.

We use the term 'surface layer' to describe the region where both air and water are present, rather than using it for a viscous boundary layer as some authors do. The main aim of the paper is to provide surface boundary conditions that are a simplification of the turbulent air–water mixture making up the surface layer such as occurs at the front and over the top of a spilling breaker. Expressions for boundary conditions are obtained by first taking the usual Reynolds ensemble averages and then integrating across the surface layer. The resulting boundary conditions include a range of new terms. The boundary conditions can best be interpreted as a coupling of two sets of terms: those that arise from normal boundary conditions for a turbulent liquid and those that represent flow in the surface layer. These boundary conditions are not entirely new since Hasselmann (1971) gave the main features: however, that paper gave much greater emphasis to waves on the water surface and the turbulent

aspects seem to have been almost entirely ignored in more recent work. The multitude of terms needed when considering boundary conditions for turbulent kinetic energy is daunting, and we have throughout simplified expressions by treating the flow as being two-dimensional. In almost all cases it is straightforward to include the extra turbulent and mean flow terms for three dimensions and we believe we have not missed any important aspects by attempting to clarify the analysis by this simplification.

All the many new terms in the boundary conditions need extra information in order to 'close' the system made up of some chosen partial differential equations modelling turbulence in the body of the liquid and these boundary conditions. For many turbulent flows there is a wealth of experimental data to assist in developing such closures, but we find that little is known about strongly disturbed free surfaces – even those which are continuous rather than broken into drops, bubbles and interpenetrating two-phase flows. One of the aims of the present papers is to stimulate experimental work by indicating what properties of the surface layer are needed.

An outline of the approximations and closure assumptions needed is given for two different examples. The simplest case we can envisage is where the surface disturbance is entirely due to turbulence reaching an otherwise horizontal surface from below. We do not give all the details but indicate how simple assumptions can lead to closure relations for basic quantities such as the equivalent surface-layer thickness d which can be related to the local turbulence intensity k and length scale. We give initial discussions and simple estimates for several of the turbulent free-surface regimes described in Part 1. In particular we note that the variation in the boundary conditions should lead to some mean flows, since in most case there are gradients of turbulent properties over the free surface. For example, Walker, Chen & Willmarth (1995) and Hong & Walker (2000) show that the spreading of a turbulent jet is substantially enhanced near a free surface. Experimental study of such mean flows may give insight into the surface layer for cases where splashing etc. makes direct measurement difficult.

The second discussion of closure for the boundary conditions is an artificial representation of a splashing surface which only takes into account the drops that are projected into free fall. Even this very simple model soon builds up to some complexity. However, it can also be approached *ab initio* and then shows the importance of the weight of the surface layer, and also the fact that if water is projected there are then some non-local influences as water can fall from one level to another gaining energy and tangential momentum on the way.

One of the important aspects of breakers that we hoped to clarify by this work is their source of energy. We hesitate to claim that the above mentioned falling of water is the full story, but it is one of the major aspects of energy production that can be identified. It seems contradictory that there is no gravity driving term in the boundary conditions for turbulent energy. However, inspection of the equation's derivation shows that the only contribution from gravity is to the mean flows in the surface layer, stressing the importance of these mean flow terms.

Finally we discuss the lower edge of a breaker, where we suggest new terminology of 'foot and toes' that stimulates what may be the most useful of the several concepts that arise in these papers. That is that the surface-layer thickness, and its mean velocity at the mean position of the breaker foot give an instantaneous indication of breaker strength at a point where much, if not most, of its turbulence is generated. These are properties that it should be possible to measure experimentally. These two measures should enable better modelling of unsteady breakers. The only previous

measure of breaker strength has been the breaker height which is then usually related to steady hydraulic jumps or bores.

During the preparation of this and the companion paper (Part 1) the authors benefitted from very useful conversations with many researchers. Among them we wish to particularly thank Professor I. A. Svendsen, Professor J. A. Battjes, Dr R. R. Kerswell and Dr C. P. Caulfield. The referees are thanked for their comments which have improved the papers presentation. Support from the European Commission Research Grants ERBCHBICT930678, MAS2-CT92-0047 and MAS3-CT97-0081 (SASME) is gratefully acknowledged. We also acknowledge support from the Office of Naval Research in its NICOP program, award no. N00014-97-1-0791 and from the Italian MURST Grant 'Processi vorticosi, turbolenti e caotici – Applicazioni impiantistiche e ambientali'.

Appendix. The equation for the convection–diffusion of a passive tracer

Breaking waves are important factors when considering convection–diffusion processes undergone by passive tracers at an air–water interface. They are especially important for their role in modelling the coupling between atmosphere and ocean large-scale dynamics where much heat, momentum and mass transfer occurs through micro-breakers (e.g. Melville 1996). As a step towards analysing exchange processes at the ocean–atmosphere interface we developed a conservation equation for a passive tracer of concentration $\mathscr{C}$ which is subject to both advection by the mean and turbulent flow fields and molecular diffusion.

The integral equation which governs the convection–diffusion process can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{V}}\langle\rho\mathscr{C}\rangle\,\mathrm{d}\mathbb{V}+\int_{\mathbb{S}}\langle\rho\mathscr{C}\mathscr{U}_j\rangle e_j\,\mathrm{d}\mathbb{S}=\int_{\mathbb{S}}\left\langle\rho\kappa\frac{\partial\mathscr{C}}{\partial x_j}\right\rangle e_j\mathrm{d}\mathbb{S}+\left\langle\int_{\mathscr{S}}\rho\kappa\frac{\partial\mathscr{C}}{\partial x_j}\tilde{n}_j\,\mathrm{d}\mathscr{S}\right\rangle \quad (A\,1)$$

in which averaging now is not restricted to the water phase and κ is the specific thermal or molecular diffusivity (constant).

We can rewrite (A 1) by using (4.3) and, after applying Green's theorem, it becomes

$$\int_{\mathbb{V}}\left[\frac{\partial\langle\rho\mathscr{C}\rangle}{\partial t}+\frac{\partial\langle\rho\mathscr{C}\mathscr{U}_j\rangle}{\partial x_j}-\frac{\partial}{\partial x_j}\left\langle\rho\kappa\frac{\partial\mathscr{C}}{\partial x_j}\right\rangle+\left\langle\rho\kappa\frac{\partial\mathscr{C}}{\partial x_j}\frac{\partial I}{\partial x_j}\right\rangle\right]\mathrm{d}\mathbb{V}=0. \quad (A\,2)$$

The following differential equation is then obtained by using constant density ρ:

$$\frac{\partial\langle\mathscr{C}\rangle}{\partial t}+\frac{\partial\langle\mathscr{C}\mathscr{U}_j\rangle}{\partial x_j}-\kappa\frac{\partial^2\langle\mathscr{C}\rangle}{\partial x_j\,\partial x_j}+\kappa\left\langle\frac{\partial\mathscr{C}}{\partial x_j}\frac{\partial I}{\partial x_j}\right\rangle=0. \quad (A\,3)$$

We subsequently rewrite the equation for our local coordinates (s,n) as

$$\frac{\partial\langle\mathscr{C}\rangle}{\partial t}+\frac{\partial\langle\mathscr{C}\mathscr{U}\rangle}{\partial s}+\frac{\partial\langle\mathscr{C}\mathscr{V}\rangle}{\partial n}-\kappa\frac{\partial^2\langle\mathscr{C}\rangle}{\partial s^2}-\kappa\frac{\partial^2\langle\mathscr{C}\rangle}{\partial n^2}+\Psi_{ss}+\Psi_{nn}=0 \quad (A\,4)$$

in which

$$\Psi_{ss}=\kappa\left\langle\frac{\partial\mathscr{C}}{\partial s}\frac{\partial I}{\partial s}\right\rangle,\quad \Psi_{nn}=\kappa\left\langle\frac{\partial\mathscr{C}}{\partial n}\frac{\partial I}{\partial n}\right\rangle$$

are the interfacial contributions involving gradients of I.

Since averaging is now extended to all flow properties (i.e. concentration and

velocity) over each phase (i.e. air and water) we can rewrite (A 4) by using (2.13) and (2.14):

$$\frac{\partial}{\partial t}[\gamma C_w + (1-\gamma)C_a] + \frac{\partial}{\partial s}[\gamma C_w U_w + (1-\gamma)C_a U_a + \gamma\langle cu\rangle_w + (1-\gamma)\langle cu\rangle_a]$$
$$+\frac{\partial}{\partial n}[\gamma C_w V_w + (1-\gamma)C_a V_a + \gamma\langle cv\rangle_w + (1-\gamma)\langle cv\rangle_a] - \kappa\frac{\partial^2}{\partial s^2}[\gamma C_w + (1-\gamma)C_a]$$
$$-\kappa\frac{\partial^2}{\partial n^2}[\gamma C_w + (1-\gamma)C_a] + \Psi_{ss} + \Psi_{nn} = 0. \qquad \text{(A 5)}$$

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